Sufficient Conditions for a Vanishing Clarke Tax - A Note

by Hans-Werner Sinn

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the tax, he may be inclined to misrepresent his true preference for the public good. Second, since the tax paid by an individual is unrelated to his personal budget constraint, it may drive him into bankruptcy and hence become impracticable. Third, in the presence of an income effect on demand, the single citizen cannot identify, let alone reveal, his demand function unless he knows the area under the others' demand function for which he has to compensate. If he is nevertheless forced to reveal a demand function before possessing the required information, he may be induced into trying some sort of strategic behavior which is likely to destroy the nice aspects of the mechanism.

In the light of these difficulties it is desirable that the revenue of the Clarke tax be small. Although it is not quite clear what "smallness" relates to in the present context, it seems plausible to focus on the share of the revenue in the total expenditure on the public good. At any rate, this is the view of Tideman and Tullock (1976, 1977) who conjectured that the share may vanish as the number of citizens approaches infinity.

While the reasoning of Tideman and Tullock hints at the possibility of a vanishing Clarke tax, it has a more or less exemplary and tentative character which obscures the generality of this result. The purpose of this note is to set out in a more detailed form sufficient conditions for a vanishing tax. These conditions are significantly weaker than those suggested in the papers by Tideman and Tullock.

II.

The characteristics of the individual preferences are of great importance for the proof of a vanishing Clarke tax. It is assumed that these preferences can be described by $P_i(x)$ which is an inverse demand function of citizen i where $x \geq 0$ is the amount of the public good. The function is continuously differentiable and is represented by a downward sloping curve which intersects with the abscissa at some finite level of x or approaches it as x goes to infinity:

$$P_i'(x) < 0, \quad \lim_{x \to \infty} P_i(x) \leq 0 \quad (i = 1, \ldots, n). \tag{1}$$

Moreover the function is characterized by a "price elasticity" of demand,

$$\eta_i \equiv \frac{\partial x}{\partial P_i} \frac{P_i}{x}, \tag{2}$$

which is bounded from below:

$$\eta_i \geq \tilde{\eta},\ \tilde{\eta} = \text{const} < 0 \qquad (i = 1, \ldots, n). \tag{3}$$

Typically, η_i is strictly negative, but the case of satiated individuals where η_i is zero or even strictly positive is not excluded.

The marginal cost of producing the public good is assumed constant and has a value of one. Accordingly, the total production cost equals the quantity of the public good, x, in numerical terms.

The Samuelson-Lindahl solution for x is x^* and is implicitly defined by the condition

$$\sum_{i=1}^{n} P_i (x^*) = 1. \tag{4}$$

It is assumed that the community is sufficiently large to ensure that this solution brings about a strictly positive quantity of the public good and is such that every citizen has a marginal willingness to pay which is bounded away from unity or, equivalently, that no one would be willing to buy the quantity x^* if he alone had to pay the total cost of the public good:

$$x^* > 0,\ P_i (x^*) \leq 1 - \varepsilon \ \text{ with } \ \varepsilon = \text{const} > 0 \tag{5}$$

$$\text{for } n \text{ sufficiently large } (i = 1, \ldots, n).$$

Let

$$P_{-i} (x) \equiv \sum_{\substack{j=1 \\ j \neq i}}^{n} P_j (x) \tag{6}$$

denote the aggregate demand function for all citizens except citizen i and let $t_i > 0$ denote this citizen's tax price, where

$$\sum_{i=1}^{n} t_i = 1. \tag{7}$$

Then, taking the tax payment from citizen i, but not respecting his preferences, the optimal quantity of the public good from the viewpoint of the other citizens, $\tilde{x}_{-i}$, is implicitly defined by

$$P_{-i} (\tilde{x}_{-i}) = 1 - t_i \qquad (i = 1, \ldots, n). \tag{8}$$

Given the inverse demand function $P_i (x)$ that was revealed by citizen i, the government authority chooses the Samuelson-Lindahl quantity x^*, and, in addition to the basic tax $t_i x^*$, citizen i is charged the amount

$$T_i = (x^* - \tilde{x}_{-i}) (1 - t_i) - \int_{\tilde{x}_{-i}}^{x^*} P_{-i} (u)\, du \qquad (i = 1, \ldots, n). \tag{9}$$

This amount is his personal Clarke tax. If he had known this procedure, citizen i would have chosen the same value of x^* since he would have tried to

$$\max_x \int_0^x P_i(u)\,du - T_i(x) - t_i x,$$

a problem whose solution ($x = x^*$) clearly satisfies the Samuelson-Lindahl condition (4). This is why there is an incentive for him to honestly reveal his true demand for the public good.

III.

The proof of a vanishing Clarke tax is based on Eq. (9). With a linear approximation of the demand function, it holds that

$$\int_{\tilde{x}_{-i}}^{x^*} P_{-i}(u)\,du \approx (x^* - \tilde{x}_{-i})\left[P_{-i}(\tilde{x}_{-i}) + \frac{1}{2} P_{-i}'(x^*)\,(x^* - \tilde{x}_{-i})\right]$$

Together with (8) this implies that (9) can be written as

$$T_i = -(x^* - \tilde{x}_{-i})^2\,\frac{1}{2} P_{-i}'(x^*),$$

an expression which, because of

$$P_{-i}(x^*) - P_{-i}(\tilde{x}_{-i}) \approx P_{-i}'(x^*)\,(x^* - \tilde{x}_{-i})$$

and

$$\begin{aligned} P_{-i}(x^*) - P_{-i}(\tilde{x}_{-i}) &= [1 - P_i(x^*)] - [1 - t_i] \\ &= t_i - P_i(x^*), \end{aligned}$$

can itself be approximated by the equation

$$T_i = -\frac{[t_i - P_i(x^*)]^2}{2\,P_{-i}'(x^*)} \qquad (i = 1, \ldots, n). \tag{10}$$

Let

$$\eta_{-i}(x) \equiv \frac{\partial x}{\partial P_{-i}}\,\frac{P_{-i}}{x} \tag{11}$$

denote the elasticity of the demand curve aggregated over all citizens except citizen i. From (2) and (6) it follows that this aggregate "price elasticity" is a weighted harmonic mean of the individual elasticities,

$$\eta_{-i}(x) = \frac{1}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{\eta_j(x)}\,\frac{P_j(x)}{P_{-i}(x)}}. \tag{12}$$

A single weight is given by the share of an individual's marginal willingness to pay, $P_j(x)$, in the aggregate marginal willingness to pay, $P_{-i}(x)$, which, because of (4), (5) and (6), is strictly positive at the Samuelson-Lindahl quantity:

$$P_{-i}(x^*) \geq \varepsilon > 0 \qquad (i=1, \ldots, n). \tag{13}$$

To find out what previous assumptions imply for the size of the elasticity at the Samuelson-Lindahl quantity it is useful to write (12) in the form

$$\eta_{-i}(x^*) = \frac{1}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{k} \frac{1}{\eta_j(x^*)} \frac{P_j(x^*)}{P_{-i}(x^*)} + \sum\limits_{\substack{j=k+1 \\ j \neq i}}^{n} \frac{1}{\eta_j(x^*)} \frac{P_j(x^*)}{P_{-i}(x^*)}} \tag{14}$$

where it is assumed, without any loss of generality, that the citizens are numbered in such a way that the first k of them have a strictly positive marginal willingness to pay $[P_j(x^*) > 0,\ \eta_j(x^*) < 0]$ and that the remaining $n-k$ ones are satiated with the public good[2] $[P_j(x^*) \leq 0,\ \eta_j(x^*) \geq 0]$. Since both of the sums in the denominator of this expression are negative, η_{-i} is negative too. Moreover, it follows from (3) that

$$\eta_{-i}(x^*) \geq \frac{1}{\frac{1}{\tilde{\eta}} \sum\limits_{\substack{j=1 \\ j \neq i}}^{k} \frac{P_j(x^*)}{P_{-i}(x^*)} + \sum\limits_{\substack{j=k+1 \\ j \neq i}}^{n} \frac{1}{\eta_j(x^*)} \frac{P_j(x^*)}{P_{-i}(x^*)}}. \tag{15}$$

If there are no satiated citizens, i. e., if $k=n$, then the first sum in the denominator of (15) is unity and the right-hand side of (15) equals $\tilde{\eta}$. If there are satiated citizens $(k<n)$, the first sum is at least unity and the second is strictly negative. Since $\tilde{\eta}$ is also strictly negative this implies that the right-hand side of (15) is strictly greater than $\tilde{\eta}$. Hence we can conclude that the aggregate price elasticity is bounded in the range

$$\tilde{\eta} \leq \eta_{-i}(x^*) < 0 \qquad (i=1, \ldots, n) \tag{16}$$

when the quantity of the public good is on the Samuelson-Lindahl level.

Utilizing the aggregate price elasticity η_{-i} and Eqs. (4) and (6), it is possible to calculate from (10) the share of citizen i's Clarke

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[2] Note that (1), (2), and (5) imply $P_j(x^*)/\eta_j(x)^* = x^*/(\partial x/\partial P_j) < 0$ when $P_j(x^*) = \eta_j(x^*) = 0$.

tax in the total expenditure on the public good:

$$\frac{T_i}{x^*} = -\frac{[t_i - P_i(x^*)]^2}{2\,P_{-i}(x^*)}\,\eta_{-i}(x^*) \qquad (i=1,\ldots,n). \tag{17}$$

Summing over all individuals yields the total share of the Clarke tax in the public expenditure,

$$\frac{T}{x^*} = -\sum_{i=1}^{n}\frac{[t_i - P_i(x^*)]^2}{2\,P_{-i}(x^*)}\,\eta_{-i}(x^*), \tag{18}$$

where $T \equiv \sum_{i=1}^{n} T_i$. Since, according to (16) and (13), there are lower bounds for $\eta_{-i}(x^*)$ and $P_{-i}(x^*)$, there is an upper bound for this share which is given by

$$\frac{T}{x^*} \leq \frac{-\tilde{\eta}}{2\,\varepsilon}\,y(n), \tag{19}$$

with

$$y(n) \equiv \sum_{i=1}^{n}[t_i - P_i(x^*)]^2. \tag{20}$$

Note that (4) and (7) imply that for the averages of the t_i's and the P_i's we have

$$\frac{\sum_{i=1}^{n} t_i}{n} = \frac{\sum_{i=1}^{n} P_i(x^*)}{n} = \frac{1}{n}.$$

This fact allows (20) to be rewritten as

$$\begin{aligned} y(n) &= \sum_{i=1}^{n}[(t_i - 1/n) - (P_i - 1/n)]^2 \\ &= n\sum_{i=1}^{n}\frac{(t_i - 1/n)^2}{n} - 2\,n\sum_{i=1}^{n}\frac{(t_i - 1/n)\,(P_i - 1/n)}{n} \\ &\quad + n\sum_{i=1}^{n}\frac{(P_i - 1/n)^2}{n} \\ &= n\,\mathrm{var}\,(t) - 2n\,\mathrm{cov}\,(t, P) + n\,\mathrm{var}\,(P), \end{aligned} \tag{21}$$

where var (u) denotes the variance of u and cov (u, v) the covariance between u and v. Let $\varrho(u, v) = \mathrm{cov}\,(u, v)/[\sqrt{\mathrm{var}\,(u)}\,\sqrt{\mathrm{var}\,(v)}]$ be the coefficient of correlation between u and v and $\gamma(u) = \sqrt{\mathrm{var}\,(u)}\,n/\sum_{i=1}^{n} u_i$ the coefficient of variation of u. Then (21) can be transformed to

$$\begin{aligned} y(n) &= \frac{n}{n^2}\,\frac{\mathrm{var}\,(t)}{(1/n)^2} - \frac{2\,n}{n^2}\,\frac{\sqrt{\mathrm{var}\,(t)\,\mathrm{var}\,(P)}}{(1/n)^2}\,\varrho(t, P) + \frac{n}{n^2}\,\frac{\mathrm{var}\,(P)}{(1/n)^2} \\ &= \frac{1}{n}\,[\gamma^2(t) - 2\,\gamma(t)\,\gamma(P)\,\varrho(t, P) + \gamma^2(P)], \end{aligned} \tag{22}$$

and (19) becomes

$$\frac{T}{x^*} \leq \frac{-\tilde{\eta}}{2\,\varepsilon}\,\frac{1}{n}\,[\gamma^2\,(t) - 2\,\gamma\,(t)\,\gamma\,(P)\,\varrho\,(t, P) + \gamma^2\,(P)]. \tag{23}$$

Expression (23) shows that the upper bound for the share of the Clarke tax in the total production cost of the public good depends crucially on the coefficient of variation of the tax price [$\gamma\,(t)$], on the coefficient of variation of the marginal willingness to pay measured at the Samuelson-Lindahl quantity [$\gamma\,(P)$], and on the coefficient of correlation between the tax price and the marginal willingness to pay [$\varrho\,(t, P)$]. Suppose the two coefficients of variation have the same value $\gamma\,(t) = \gamma\,(P) \equiv \gamma$. Then (23) reduces to

$$\frac{T}{x^*} \leq \frac{-\tilde{\eta}}{n\,\varepsilon}\,[1 - \varrho\,(t, P)]\,\gamma^2,$$

and it is clear that T/x^* approaches zero as ϱ goes to unity. Thus, the better the tax authority succeeds in finding the Lindahl prices, the lower is the revenue of the Clarke tax. The main difficulty in the allocation of public goods is, however, that the tax authority is unable to assess Lindahl prices. Thus, the interesting question is whether or not the Clarke tax will vanish with an increase in the size of the community even if the coefficient of correlation between the tax price and the marginal willingness to pay is bounded away from unity.

The answer to this question follows immediately from (23). Since $-1 \leq \varrho \leq +1$ and since T/x^* cannot be negative it obviously holds that

$$\lim_{n \to \infty} \frac{T}{x^*} = 0 \tag{24}$$

if

$$\frac{\gamma\,(t\,(n))}{\sqrt{n}}, \frac{\gamma\,(P\,(n))}{\sqrt{n}} \to 0 \quad \text{for} \quad n \to \infty. \tag{25}$$

In words, but slightly less precisely, this can be expressed as follows.

Proposition: With demand functions as specified above, the share of the Clarke tax in the total expenditure of the public vanishes with an increase in the population of the community if both the coefficient of variation of the marginal willingness to pay measured at the Samuelson-Lindahl quantity and the coefficient of variation of the tax price rise strictly less than proportionately to the square root of the quantity of the population.

If we find it plausible that the coefficients of variation stay constant despite a rise in n, i. e. that by and large the relative deviations of the t_i's and P_i's from their common mean $1/n$ stay constant, then this proposition clearly proves the robustness of the Tideman-Tullock conjecture. However, not only with constant relative deviations, but even with a modest rise in these deviations, there are strong forces which drive the revenue share of the Clarke tax to zero as the size of the community increases.

Casual readers tend to interpret the remarks of Tideman and Tullock in such a way that a vanishing Clarke tax requires the absolute difference between the tax price and the marginal willingness to pay to be less than $1/n$ for each citizen[3]. This strong limitation is sufficient, but fortunately it is not necessary for the result. For a person who pays the average tax price, but has a high preference for the public good, it may well be the case that $P_i - t_i \geq 1/n$. And $t_i - P_i \geq 1/n$ may occur since there are people who, despite a normal preference for the public good, are assigned a high tax price or who pay the average tax price but are satiated with the public good in the sense that $P_i(x^*) < 0$. These possibilities are perfectly compatible with a vanishing Clarke tax. It is the statistical properties of the aggregate that matter for the result, and not the individual exceptions.

References

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Address of author: Prof. Dr. Hans-Werner Sinn, Volkswirtschaftliches Institut der Universität München, Ludwigstrasse 33, D-8000 München 22, Federal Republic of Germany.

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[3] Cf. in particular Tideman and Tullock (1977, p. 126).