

Expected Utility and the Siegel Paradox

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The Siegel paradox and related phenomena have been discussed by Roper (1975), McCulloch (1975), Flemming, Turnovsky, and Kemp (1977), Boyer (1977), and others. These authors did not reach a final conclusion about the theoretical significance of the paradox. However, there seems to have been broad agreement on its limited practical relevance. Even Siegel (1975, p. 175) himself conceded that the profits his paradox implied were "empirically insignificant". He accepted McCulloch's view that they were too small to be tested empirically with the available exchange rate data and he seemed to believe that this implied that the profits were also too small to explain speculation.

The present paper challenges this view. It does not go into the deep issues of whether or not Siegel type profits are possible for *all* members of a society and whether these profits reflect net welfare gains. For this a stochastic general equilibrium model would have to be constructed.¹ Nor does the paper offer better empirical data than McCulloch believed to be available. In the author's opinion McCulloch's argument does not lead very far. It demonstrates the difficulties of testing the validity of mathematical theorems empirically. However, it certainly does not in itself imply that the Siegel type profit is too small to induce a significant speculative commitment. After all, there appears to be no theoretical reason for arguing that speculators mistrust the axioms of mathematics and behave as if they based their decisions on econometric tests only. Instead, there is good reason for assuming that rational speculators will try to satisfy the von Neumann-Morgenstern axioms and behave as if they were maximizing expected utility.

A more relevant question for judging the empirical significance of the paradox is therefore how much of their wealth are expected utility maximizers willing to risk for Siegel type profits? This question is discussed in this paper. Using plausible utility functions and observations on the size of risk aversion in other choice problems under uncertainty, the paper attempts to predict the decision of an agent who is confronted with the choice between putting his total wealth at stake for the Siegel profit and remaining uncommitted. It is true that this is an extreme choice situation for, even if the Siegel profit induces some speculation, it seems very unlikely a priori that anyone would accept a bet that involves staking *everything* he owns. However, if it nevertheless turned out that the agent prefers to speculate, then there would clearly be

¹ See Kemp and Sinn (1989).

good reason for rejecting the view that the Siegel paradox is negligible. The Siegel paradox would then have to be considered as one of the important explanations for speculation.

2. Basic Assumptions

To demonstrate the Siegel paradox and to investigate the strength of the incentives to speculate it generates, consider the example of foreign exchange speculation more closely. There are two countries, America and Germany, and the price of deutschmarks in terms of dollars at time t is x_t . In each country, there are various agents participating in foreign exchange contracts, but attention is focussed on speculators who consume only their respective domestically produced commodities. The prices in domestic currency of these commodities are assumed constant and equal to unity. American speculators therefore derive their utility from the dollars and German speculators from the deutschmarks they gain.

Speculators believe the real exchange rate to be generated through a stationary stochastic process of the type

$$x_t = \bar{x}v_t. \quad (1)$$

Here \bar{x} is a strictly positive constant and $v_t, v_t > 0$, a multiplicative stochastic disturbance term which is free of serial correlation. The variables x_t and v_t are variates of the corresponding random variables X_t and V_t . The distribution of V is time-invariant and has a strictly positive variance; otherwise it can have arbitrary properties. Notice that the constant \bar{x} is *not* the expected value of X and that it is *not* required that $E(V) = 1$.

It is assumed that investment in each country offers the same non-random rate of return measured in terms of the respective currency. This assumption implies that covered interest arbitrage strictly links the forward rate x_t^f , quoted at time t for time $t+1$, with the current spot rate:

$$x_t^f = x_t. \quad (2)$$

The assumption is not essential. Its only purpose is to eliminate any obvious reason for speculative profits.

The previous assumptions are symmetrical with respect to both alternative definitions of the exchange rate. In particular, the development of the DM/\$ rate $1/x_t$ is also described by a stochastic process with a horizontal trend ($1/\bar{x}$) and a stochastic

disturbance term $(1/v_t)$ free of serial correlation. For this reason, all of the conclusions derived in this paper will simultaneously hold for speculators from both countries.

Consider an American speculator who, at time t , plans to spend his total wealth of z_t dollars (which he will possess after collecting interest at time $t+1$) on deutschmark futures and to sell them immediately on receipt at the predetermined date $t+1$. If this speculator were endowed with divine foresight, he would know that he will end up with wealth of amount $z_t x_{t+1}/x_t^f = z_t x_{t+1}/x_t = z_t v_{t+1}/v_t$, and obviously the commitment would then be profitable for him if, and only if, $v_{t+1}/v_t > 1$. The problem, however, is to define the conditions under which an imperfectly informed speculator would be willing to engage in the forward contract.

The paper distinguishes between three alternative types of speculator each of which differs from the others with regard to the subjective degree of ignorance. Speculators of type I and II do not know \bar{x} and therefore are unable to compute v_t from the observation of x_t . Provided that they find the forward contract profitable at all and know (1), they would not object to committing themselves before time t ; i. e., before observing the forward rate x_t^f . The difference between type-I and type-II speculators concerns their knowledge about the distribution of V . While the type-I speculator feels certain about this distribution, the type-II speculator considers various distributions possible and is unable to decide which distribution he should assume. The speculator of type III feels sure about both the size of \bar{x} and the shape of the probability distribution of V . He is able to calculate v_t from observing x_t .

3. Uninformed Speculators

Consider first a type-I speculator who does not know \bar{x} but knows the distribution of V . If he invests his total wealth z_t , the (random) level of final wealth Z_{t+1} is given by

$$Z_{t+1} = z_t (V_{t+1}/V_t). \quad (3)$$

Since, by assumption, V_{t+1} and V_t are stochastically independent, V_{t+1} and $1/V_t$ share this property and hence $E(Z_{t+1}) = z_t E(V) E(1/V)$ or

$$E(Z_{t+1}) = z_t [E(V)/H(V)] > z_t, \quad (4)$$

where $H(V) \equiv 1/E(1/V)$ is the harmonic mean of V . The mathe-

mathematical fact that $H(V) < E(V)$ if $V > 0$ and if $\text{var}(V) > 0$ can be seen as the essence of the Siegel paradox (although Siegel did not consider the case where V_t or X_t is stochastic). It ensures that $E(Z_{t+1}) > z_t$. Thus, even a fairly ignorant speculator will expect a profit from investing his wealth in selling his own currency short in the forward market.

The question to be solved in this paper is whether this expected profit is sufficiently large to compensate for risk aversion when the decision maker is an expected utility maximizer. In principle, it can be answered by calculating the certainty equivalent of final wealth. If this certainty equivalent exceeds initial wealth, the speculator is willing to engage in the contract and to risk his wealth. If it falls short of initial wealth, he rejects the contract. If it equals the initial wealth, he is indifferent between risking his wealth and staying uncommitted.

An expected utility maximizer's certainty equivalent of the distribution Z_{t+1} is generally defined as $S(Z_{t+1}) = U^{-1} \{ E[U(Z_{t+1})] \}$ where $U(\cdot)$ is his von Neumann-Morgenstern utility-of-wealth function. Consider the class of utility functions that exhibit constant and strictly positive relative risk aversion. $U(z_{t+1}) = (1 - \varepsilon) z_{t+1}^{1-\varepsilon}$ for $0 < \varepsilon \neq 1$ and $U(z_{t+1}) = \ln z_{t+1}$ for $\varepsilon = 1$ where ε is the Pratt-Arrow measure of relative risk aversion. Then the certainty equivalent is²

$$S(Z_{t+1}) = \begin{cases} [E(Z_{t+1}^{1-\varepsilon})]^{1/(1-\varepsilon)} & \text{for } 0 < \varepsilon \neq 1 \\ \exp E(\ln Z_{t+1}) & \text{for } \varepsilon = 1. \end{cases} \tag{5}$$

The class of constant relative risk aversion utility functions is an empirically plausible class. The class was derived from experimentally founded psychophysical laws by Sinn (1983, ch. III; 1985) and it contains as a special case the logarithmic utility function which Bernoulli (1713) favored and which Arrow (1970, p. 98) considered as an admissible first-order approximation to a utility function that satisfies his boundedness axioms. Constant relative risk aversion implies that, when a person becomes richer and continues to invest all his wealth in the same asset, the certainty equivalent remains in fixed proportion to the expected level of final wealth.

Consider first the case $\varepsilon \neq 1$. Inserting (3) into (5) gives

$$S(Z_{t+1}) = z_t [E(V_{t+1}^{1-\varepsilon}/V_t^{1-\varepsilon})]^{1/(1-\varepsilon)} \quad \text{for } \varepsilon \neq 1. \tag{6}$$

² Cf. Sinn (1983, pp. 149 ff.).

Notice that the stochastic independence of V_{t+1} and V_t implies a stochastic independence of $V_{t+1}^{1-\varepsilon}$ and $V_t^{1-\varepsilon}$. This ensures that, as in the step from (3) to (4), (6) can be transformed to

$$S(Z_{t+1}) = z_t [E(V^{1-\varepsilon})/H(V^{1-\varepsilon})]^{1/(1-\varepsilon)} \{ \geq \} z_t \text{ for } \varepsilon \{ \leq \} 1. \quad (7)$$

Obviously, since $E(\cdot)/H(\cdot) > 1$ and $1/(1-\varepsilon) \{ \geq \} 0$ as $\varepsilon \{ \leq \} 1$, the certainty equivalent exceeds initial wealth if $\varepsilon < 1$, but falls short of it if $\varepsilon > 1$.

Consider next the case $\varepsilon = 1$. Here (3) and (5) imply

$$S(Z_{t+1}) = z_t \exp E(\ln V_{t+1} - \ln V_t) \quad \text{for } \varepsilon = 1.$$

Since $E(\ln V_{t+1} - \ln V_t) = E(\ln V_{t+1}) - E(\ln V_t) = 0$ and $\exp 0 = 1$, this gives

$$S(Z_{t+1}) = z_t \quad \text{for } \varepsilon = 1. \quad (8)$$

So far, only a speculator of type I, who feels sure about the distribution of V but does not know \bar{x} , has been considered. The speculator of type II who neither knows \bar{x} nor the correct distribution of V will carry out calculations (7) and (8) for all the alternative distributions of V he considers possible. Provided that, as is assumed, the certainty equivalents exist,³ the equality sign in (8) and the inequality signs in (7), which depend only on the parameters of the utility function, will hold for all distributions considered. Thus the speculator of type II will come to the same decision as the speculator of type I despite his higher degree of subjective ignorance.

Because of the symmetric specification of the model and the arbitrariness of the distribution of V , all of the preceding arguments hold when x_t is replaced by $1/x_t$; i. e., when the problem is seen from the point of view of the German speculator. Instead of (1), the German speculator is concerned with a stochastic process of the type $1/x_t = (1/\bar{x})(1/v_t)$ where (2) continues to be true. Obviously, this stochastic process is of the same nature as (1). The strictly positive constant \bar{x} is replaced by $1/\bar{x}$ and the multiplicative stochastic disturbance term v_t , $v_t > 0$, is replaced by $1/v_t$, $1/v_t > 0$. As the distribution of V was assumed to be time-invariant and to have strictly positive variance, the distribution of $1/V$ shares these properties; and as the distribution of V was allowed to

³ A sufficient, but not a necessary condition for existence is that V is bounded away from zero, that $U(v)$ is concave (i. e., $\varepsilon > 0$), and that $E(V)$ exists.

have an arbitrary shape all qualitative results derived for the American speculator will simultaneously be true for the German speculator. Even though the speculators of the two countries may have identical beliefs about the distribution of x , they behave like mirror images, putting their respective wealth at stake for the expected profit which the Siegel paradox simultaneously offers to each of them.⁴ The following proposition summarizes this result.

Proposition 1:

Suppose that investments offer the same non-random rate of return in each country and that speculators believe that the exchange rate follows a stationary trend with a serially uncorrelated disturbance term, but are unaware of what the trend is and, possibly, of the precise distribution of the disturbance term. Suppose further that utility depends only on the final amount of the respective domestic currency a speculator has and that relative risk aversion is less than unity. Then, even with identical expectations, speculators from both countries will strictly prefer to invest all of their wealth in forward sales of domestic currency rather than remain uncommitted. If the degree of relative risk aversion is above unity, the speculators are not willing to risk all of their wealth, and if it is unity, they are indifferent between risking all of their wealth and remaining uncommitted.

4. The Informed Speculator

Finally, let us consider the decisions of the speculator of type III, who knows equation (1), knows \bar{x} , and can calculate the variate v_t of the disturbance term from observing the current exchange rate x_t . His final wealth is given by

$$Z_{t+1} = z_t (V_{t+1}/v_t), \quad (9)$$

⁴ It is true that the American speculator makes a loss when the German makes a profit and vice versa. Thus, ex post, they cannot both gain from taking short positions in their domestic currencies. However, for each of them the potential losses are smaller than the potential gains. Suppose, for example, that $x_t = 1\$$ and that $x_{t+1} = 2\$$ or $x_{t+1} = 1/2\$$ with probability $1/2$ each. Then the expected profit per unit of domestic currency staked is

$$\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{1}{2} - 1 = \frac{1}{4} > 0$$

for both the American and the German speculator.

and the corresponding mathematical expectation is

$$E(Z_{t+1}) = z_t [E(V)/v_t].$$

In the absence of risk aversion, an American speculator of type III would risk his wealth by selling his own currency short if $E(V) \geq v_t$. Because of the symmetry, a German speculator would behave analogously if $E(1/V) \geq 1/v_t$; i. e., if $v_t \geq H(V)$. Thus, with identical expectations, risk neutral speculators will simultaneously sell their own currencies in the forward market if the variate of the disturbance term is in the range $E(V) \geq v_t \geq H(V)$. Now the question is, by how much does this range shrink in the presence of risk aversion assuming the same utility functions as before, and indeed is there such a range at all?

From (5) and (9) it follows that

$$S(Z_{t+1}) = \begin{cases} z_t \cdot (1/v_t) \cdot [E(V^{1-\varepsilon})]^{1/(1-\varepsilon)} & \text{for } 0 < \varepsilon \neq 1 \\ z_t \cdot (1/v_t) \cdot \exp E(\ln V) & \text{for } \varepsilon = 1. \end{cases}$$

In the case $\varepsilon = 1$ the American speculator of type III is willing to risk his total wealth if $\exp E(\ln V) \geq v_t$; i. e., if the observed variate of the disturbance term, v_t , falls short of the geometric mean of its distribution. Equivalently, the German speculator risks his wealth if $\exp E[\ln(1/V)] \geq 1/v_t$. Since one over the geometric mean of a random variable equals the geometric mean of its inverse — i. e., since $1/[\exp E(\ln V)] = \exp E[\ln(1/V)]$ — the condition for an engagement of the German speculator is that the observed variate of the disturbance term is not below its geometric mean: $v_t \geq \exp E(\ln V)$. Hence, with identical expectations, only at the point $v_t = \exp E(\ln V)$ can we observe a German and an American speculator of type III simultaneously risking their wealth.

In the case $\varepsilon \neq 1$ the condition of the American speculator risking his wealth is $v_t \leq [E(V^{1-\varepsilon})]^{1/(1-\varepsilon)}$ and the corresponding condition for the German speculator is $1/v_t \leq [E(1/V^{1-\varepsilon})]^{1/(1-\varepsilon)}$ or, equivalently, $v_t \geq [H(V^{1-\varepsilon})]^{1/(1-\varepsilon)}$. Thus the range where the speculators are simultaneously willing to risk their wealth is $[E(V^{1-\varepsilon})]^{1/(1-\varepsilon)} \geq v_t \geq [H(V^{1-\varepsilon})]^{1/(1-\varepsilon)}$. Since $E(\cdot) > H(\cdot)$, this range obviously exists if, and only if, $\varepsilon < 1$.

With reference to the forward rate rather than to the observed variate of the disturbance term of the current spot rate, both of which are linked through (1) and (2), these results can be summarized as follows.

Proposition 2:

Suppose the assumptions underlying Proposition 1 are changed so that the speculators have a firm knowledge of the trend of the exchange rate and the distribution of its disturbance term. Then, with identical expectations, if and only if relative risk aversion is less than unity will there be a non-degenerate range of the forward rate where the German and the American speculators are simultaneously willing to risk all of their wealth by selling their own currencies short in the forward market. If relative risk aversion equals one, this range shrinks to a single point whose value is given by the geometric mean of the future exchange rate. With relative risk aversion above unity no range exists where the speculators from both countries are simultaneously willing to risk all of their wealth.

5. Discussion

The above analysis shows that the degree of relative risk aversion is very important for the quantitative significance of the Siegel paradox. The borderline degree of relative risk aversion is unity, a value which characterizes the logarithmic utility function so emphatically favored by Bernoulli. With a higher degree, "paradoxes" in the extreme sense that speculators from both countries are willing to risk all of their wealth do not exist. However, when relative risk aversion is unity or less than unity, the risk premium implied by the Siegel paradox is large enough to make speculation so attractive that the speculators would not mind staking everything they own.

Little direct empirical evidence about the degree of relative risk aversion is available. But there are at least two arguments that suggest a degree below unity. The first is that utility of wealth is bounded from below. Utility does not approach minus infinity as wealth goes to zero since, if this were the case, no one would be willing to engage in activities with a strictly positive, yet arbitrarily small probability of ruin. Utility functions that are bounded from below have a relative risk aversion below unity (Arrow, 1970, pp. 90–120). There is no equally strong argument for utility to be bounded from above. The St. Petersburg paradox which Arrow (1970) used to defend a boundedness from above is a weak argument. It involves a game with unlimited pay-offs and creates existence problems that are of little relevance for real decision

problems. Since real games have pay-offs that are bounded from above, there is no need for utility to be bounded from above.⁵

The second argument refers to the age dependence of risk aversion. It has been frequently observed that people become more risk averse as they grow older. If this behavior is to be compatible with dynamic optimization under constant stochastic returns to scale and with preferences that satisfy the Weber-Fechner law of psychophysics, then it requires relative risk aversion to fall short of unity (Sinn, 1983, ch. IVB). Relative risk aversion above unity implies that old people are less risk averse than young people and a value of unity implies that the degree of relative risk aversion is independent of a person's age.

Of course, these arguments are not ultimately compelling reasons for excluding the case where relative risk aversion exceeds unity. However they show that the contention of a negligible Siegel effect conflicts with established implications of expected utility theory. Under plausible assumptions for the utility function the risk premium implicit in the Siegel paradox is so strong that people should be willing to risk all of their wealth for it.

It should be stressed that a degree of relative risk aversion above unity merely excludes the possibility of speculators being willing to stake *all* of their wealth. It does not imply that they would object to any commitment at all. In fact, it is well known that there is no risk aversion "in the small"; i. e., that the required risk premium per unit of standard deviation vanishes if the latter goes to zero. This implies that even a person with a degree of risk aversion above unity finds it attractive to participate in a contract that puts some, albeit not all, of his wealth at stake. In this weak sense, the Siegel paradox is always present.

In the light of these considerations, it seems that the Siegel paradox remains a puzzle whose appropriate role in economic theory has yet to be defined. One of the problems that still awaits a solution is that, contrary to the results derived, most people are not, in fact, engaged in forward speculation. For them, the risk premium implicit in the paradox is obviously not enough to make even a small speculative commitment attractive. Hesitating to leave the firm ground of expected utility theory, the theorist will tend to make transactions costs responsible for this observation. For poor people whose potential commitment is small these should be particularly important. However, the transactions costs expla-

⁵ See Sinn (1983, pp. 186—194) for a detailed discussion.

nation loses much of its force when confronted with the result that in the likely case of relative risk aversion below unity it pays to stake one's *total* wealth. Under the plausible assumption that transactions costs rise less than in proportion to the wealth committed to speculation, sufficiently rich people should be able to neglect them. This prediction is confirmed by the casual observation that most speculators belong to the upper income groups. However, it remains unclear why not everyone who is rich speculates.

There is one empirical observation that confirms the predictions of the paradox: there are always many speculators who speculate in opposite directions. It is, of course, possible to explain this by different standards of information. However, the international communication channels today are sufficiently well developed to call the general validity of this explanation into question. With identical beliefs about the exchange rate distribution, the only explanation left may be the Siegel paradox.

The Siegel paradox has unfavorable implications for the so-called efficient-markets literature that has grown so rapidly in recent years. To the extent that this literature refers to the foreign exchange market, its central hypothesis is that the actions of well-informed speculators link the forward exchange rate to the expected value of the future spot rate, and many authors have taken the unsuccessful attempts to empirically reject this hypothesis to be a sufficient argument in its favor. There seem to be severe problems here. On the one hand, in the light of McCulloch's argument, cited above, it is not surprising that the efficient-markets hypothesis has not yet been empirically rejected. On the other hand, there is the logical problem that the forward rate cannot be both an unbiased predictor of the expected spot rate and its expected inverse. The results of this paper show that, despite the empirical problems in detecting its existence, the systematic bias between the forward rate and the expected future spot rate that follows on logical grounds alone should be large enough to provide a strong stimulus for a speculative commitment.

Perhaps the clearest evidence for this possibility is that "blind" speculation around a horizontal trend (Proposition 1) is extremely attractive even for risk averse agents. Under weak assumptions, such speculation is like a siren song that entices the agents to throw caution to the winds and, if necessary, hazard their all. This result is in striking contrast to the efficient-markets proposition that, unless non-public information is available, speculation does not pay. Provided the numeraires in which speculators calculate

their profits are different, speculation does pay even for outsiders who occasionally try their luck.

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