# The Bloos Rule

Hans-Werner Sinn

From *“Economic Decisions under Uncertainty”*, Amsterdam, New York and Oxford 1983, 163-182.

# ECONOMIC DECISIONS UNDER UNCERTAINTY

HANS-WERNER SINN
*UNIVERSITY OF MANNHEIM*

1983
NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM · NEW YORK · OXFORD

*ISBN 0 444 86387 7*

First published as: *Ökonomische Entscheidungen bei Ungewißheit* in: Die Einheit der Gesellschaftswissenschaften, ed. by E. Böttcher, J.C.B. Mohr (Paul Siebeck), Tübingen 1980.

PUBLISHERS:

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM – NEW YORK – OXFORD

DISTRIBUTORS FOR THE U.S.A. AND CANADA:

ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 VANDERBILT AVENUE
NEW YORK, N.Y. 10017

PRINTED IN THE NETHERLANDS

aversion. For almost arbitrary distribution classes but small dispersions, these functions imply a homothetic indifference-curve system in the $(\mu, \sigma)$ diagram. The quality of approximation in this diagram is a function of the coefficient of variation $\sigma/\mu$ of the end-of-period wealth distribution. For distributions from a linear class whose standardized distribution to the left is bounded at $z=-\underline{k}$ there exist indifference curves in the $(\mu, \sigma)$ diagram in the range where $\mu/\sigma > \underline{k}$. These curves are an exact representation of a von Neumann-Morgenstern function: they are homothetic, convex, and enter the ordinate perpendicularly. Important implications for risk evaluation are that the intensity of insurance demand

- rises with risk aversion as measured by $\varepsilon$,
- is independent of wealth in the case of wealth insurance,
- decreases with a rise in wealth if the risk to be insured is given.

## Section B
## The Bloos Rule

In the preceding analysis it was assumed that the range of dispersion of a probability distribution to be evaluated does not exceed the range over which the Weber functions (A 34) are defined. To avoid the possibility of negative variates of wealth, distributions bounded to the left at $\upsilon=\mu-\underline{k}\sigma$ were excluded when $\mu/\sigma<\underline{k}$, $\varepsilon<1$, and when $\mu/\sigma\leq\underline{k}$, $\varepsilon\geq 1$. Moreover, distributions not bounded to the left were generally disregarded.

This exclusion seems very restrictive since among the ones it rules out is the normal distribution which, because of its approximation property for sum variables, has a significant practical relevance. On the other hand, it should not be forgotten that such an approximation, though useful, has its limitations. However similar the distributions that occur in reality seem to be to the normal distribution, in at least one respect there is a significant difference: *actual* wealth cannot become negative, because, quite clearly, no one can lose more than he has. This fact is graphically stated in the phrase 'you can't get blood out of a stone' or, to coin a word, in the 'Bloos rule'. It is true that there are many people who burden themselves with more debt than they can ever hope to repay in their lifetimes, i.e., people whose economic balance sheets, including human capital, indicate negative wealth. However, since the debtor's prison has been abolished, the fact that part of the debt is not redeemable does not worry them[1,2,3].

Let $V^n$ denote the actual or *net* distribution of wealth and let $V$ denote the balance-sheet or *gross* distribution of wealth. Then the BLOOS rule is

$$V^n = \begin{cases} V, & V \geq 0, \\ 0, & V \leq 0. \end{cases} \tag{1}$$

Given the Weber functions this relationship implies a complete preference ordering over gross distributions whose properties will be studied in what follows. It will be useful to carry out this study separately for the cases of weak ($\varepsilon < 1$) and strong ($\varepsilon \leq 1$) risk aversion, since, in the first case the utility function is bounded from below, while in the second it is not.

## 1. *The Complete Preference Ordering under Weak Risk Aversion ($0 < \varepsilon < 1$): The True Reason for Risk Loving*

### 1.1. *The Derived Utility Function for Gross Wealth Distributions*

Using (1) and letting the Weber functions (A 34) be denoted by $U^n(.)$ we can construct the following *derived* utility function $U(.)$:

$$U(\upsilon) = \begin{cases} U^n(\upsilon) = \upsilon^{1-\varepsilon}, & \upsilon \geq 0, \\ U^n(0) = 0, & \upsilon \leq 0, \end{cases} \tag{2}$$

with

$$U'(\upsilon) = (1-\varepsilon)\upsilon^{-\varepsilon} \quad \text{and} \quad U''(\upsilon) = -\varepsilon(1-\varepsilon)\upsilon^{-(1+\varepsilon)}, \quad \text{if } \upsilon > 0,$$

and with

$$U'(\upsilon) = U''(\upsilon) = 0, \quad \text{if } \upsilon < 0.$$

---

[1] Thus a formal test developed by SCHNEEWEISS (1964; 1967a, pp. 129–160) for finding out the intersection of preference structures that on the class of normal distributions, may just as well be represented in a $(\mu, \sigma)$ diagram as by means of a von Neumann-Morgenstern function cannot be applied. One of the conditions for the application of this test, $\lim_{\upsilon \to -\infty} U(\upsilon)e^{-\alpha\upsilon^2} > -\infty$, $\alpha = \text{const.} > 0$, cf. (1967a, p. 131), is not satisfied.

[2] Recall the definition of wealth given at the beginning of this chapter. According to this definition a complete loss of wealth means that, during the whole of his life, the decision maker still retains enough for subsistence minimum consumption. Because of the limits to attachment usual in countries with a well-developed law system this seems to be a realistic assumption.

[3] The significance of a lower boundary of wealth for the evaluation of risks was also pointed out by SEIDL (1972, pp. 443–445). Seidl did not attempt to integrate this boundary into a formal decision theoretic approach. The following analysis extends the one given in SINN (1982) by considering arbitrary distribution classes rather than binary distributions only.

Since the derived utility function $U(.)$ evaluates the gross distribution $V$ in exactly the same way as the original function $U^n(.)$ evaluates the net distribution $V^n$ we have

$$(3) \qquad E[U(V)] = E[U^n(V^n)]$$

for all elements of the opportunity set.

$U(.)$ is nothing more then an auxiliary mathematical construct that draws all its information from $U^n(.)$. Thus we should not be bothered by the fact that $U(v)$ does not satisfy the Non-Saturation Axiom for $v \leq 0$. $U^n(v^n)$ is compatible with this axiom for all admissible values of $v^n$ and that is sufficient.

By analogy with Figure 10 in section II C 1.2, the following Figure 9 demonstrates the implications of the Bloos rule for the intensity of insurance demand if, because of $l > aq$, the loss distribution $C = \begin{pmatrix} w & 1-w \\ l & 0 \end{pmatrix}$ is large enough to allow for negative gross wealth, a case that is particularly relevant in the case of liability insurance.

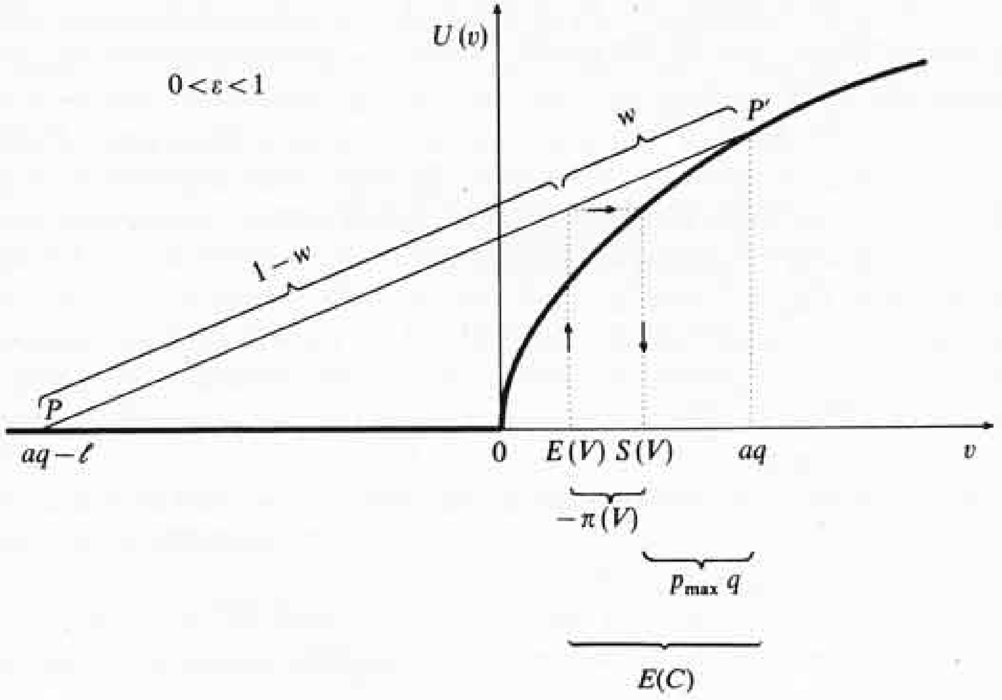


Figure 9

The remarkable aspect of this figure is that the BLOOS rule implies a kink in the derived utility function that destroys the over-all concavity that is usually assumed. If the possible loss $l$ is large enough, the certainty equivalent $S(V)$ exceeds the expected level of wealth so that the subjective price of risk $\pi(V)$ is negative and the intensity of insurance demand is smaller than one:

$$(4) \qquad g = \frac{p_{\max} q}{E(C)} < 1.$$

One implication for the case of liability insurance is obvious. Since insurance companies have to demand a premium at least equal to the expected indemnification payment[4], it is preferable for the person facing the liability risk to stay uninsured, although, concerning his subjective preferences, he is a risk averter in the usual sense. The reason is that, even without insurance, the person liable avoids part of the loss which, of necessity, is borne by other parties sustaining the damage. Insurance is not attractive since part of the premium that the buyer pays benefits these other parties and not himself.

Apart from the insurance example that will be considered in more detail later[5], there are a number of other significant implications of the BLOOS rule. For example, the rule suggests that, when choosing between different techniques of production, a firm may well decide in favor of extremely risky techniques that involve the possibility of losses far exceeding the value of its equity, simply because a large part of these losses would be borne by others. In this case, the implication of the BLOOS rule that there will be negative external effects, which may result in a substantial misallocation of resources, is straightforward. Another implication that will also be discussed in detail[6], shows up in forward speculation when speculators sell short on the futures market, where the possible loss may greatly exceed the speculator's wealth. According to the BLOOS rule, in this case it may well be rational for a risk averter, when choosing between two contracts, to decide on the one with the lower expected gain and the higher variance with respect to gross variables. The BLOOS rule may therefore explain why speculators, in particular, are often said to be risk lovers.

These remarks concerning the practical relevance of the kinked utility function for evaluating gross wealth distributions should be enough for the time being.

---

[4] Cf. chapter II C 1.2 and 1.3.

[5] In chapter V C 1 and V C 2.3.

[6] In chapter V B 4.

### 1.2. *Indifference Curves in the $(\mu, \sigma)$ Diagram for Linear Distribution Classes*

This section investigates the implications of the Bloos rule for the shape of the indifference curves in a $(\mu, \sigma)$ diagram in order to facilitate an evaluation of distributions other than the binary distribution considered above. For the sake of simplicity, the analysis is confined to probability distributions that are described by continuous density functions[7]. Moreover we only consider distributions from the same arbitrary linear class[8] with $f_z(z;0,1) \equiv f_z(z)$ and $Z \equiv (V-\mu)/\sigma$ or $z \equiv (v-\mu)/\sigma$, respectively, where $z$ is a variate of the standardized random variable $Z$. For the time being the distributions are assumed to be bounded to the left at $z=-\underline{k}$, $\underline{k} \leq \infty$, and to the right at $z=\bar{k}$, $\bar{k} \leq \infty$. It is assumed that the density is finite except possibly for $z=\bar{k}$.

This characterization of the linear distribution class has the following immediate consequence. Above a ray through the origin $\mu=-\bar{k}\sigma$ (cf. Figure 10) there is a continuum of indifference curves. They all enter the positive part of the $\mu$ axis because each distribution, which brings about strictly positive wealth levels with a probability greater than zero, has a strictly positive certainty equivalent. Since the net distribution is bounded at $v=0$, the lowest conceivable certainty equivalent is zero. Below the border line $\mu=-\bar{k}\sigma$ there is an indifference area. All distributions whose mappings are in this area only extend over the negative half of the wealth axis where $U(v)=\text{const.}$ Thus these distributions bring about a non-random net wealth level of zero and hence the decision maker regards them as equal in value.

Because of the continuity of $U(v)$ in the whole range $-\infty \leq v \leq +\infty$, the indifference-curve slope can, in the usual way, be calculated by use of equation (II D 60)[9]. An interesting question is whether the result of homothetic indifference curves given by equation (III A 48) remains valid. This question can be answered in the affirmative since the equation $U'(\mu+\sigma z)=(1-\varepsilon)\sigma^{-\varepsilon}U'(z+\mu/\sigma)$, that was implicitly used in

---

[7] If there are discrete distributions they may be approximated by a continuous one. Cf. footnote 2 in the introduction to chapter II.

[8] The method of local approximation can no longer be applied since the Weber functions can only be developed into a polynomial down to 50% of the mean. Cf. chapter II D 2.2.2.

[9] The necessary differentiation under the integral

$$\int_{-\infty}^{+\infty} f_z(z)U(\mu+\sigma z)\,dz$$

does not alter the formula despite the discontinuity in the marginal utility function at $v=0$. Cf. appendix 3 to this chapter where, for another problem, the same mathematical aspect shows up. Formula (5) would cease to hold only if $U(v)$ were discontinuous.

the derivation of (III A 48), not only holds for $\mu + \sigma z > 0$, but also for $\mu + \sigma z < 0$ since in this case, from (2), $U'(\mu + \sigma z) = 0$. Thus the formula

$$\text{(5)} \qquad \left.\frac{d\mu}{d\sigma}\right|_{U(\mu,\sigma)} = -\frac{E\left[ZU'\left(\frac{\mu}{\sigma} + Z\right)\right]}{E\left[U'\left(\frac{\mu}{\sigma} + Z\right)\right]}$$

remains valid in the case of gross wealth distributions with negative variates so that the indifference-curve system is also homothetic in the range below or to the right of the curve $\mu = \underline{k}\sigma$ (cf. Figures 6 and 10).

Over some range, the indifference curves plotted in Figure 10 have negative slopes, which is a sign of risk loving behavior. This property is already necessitated by the fact that *all* distributions with $\mu/\sigma > -\bar{k}$ have strictly positive certainty equivalents and is plausible in the light of the convexity of the utility function brought about by the BLOOS rule.

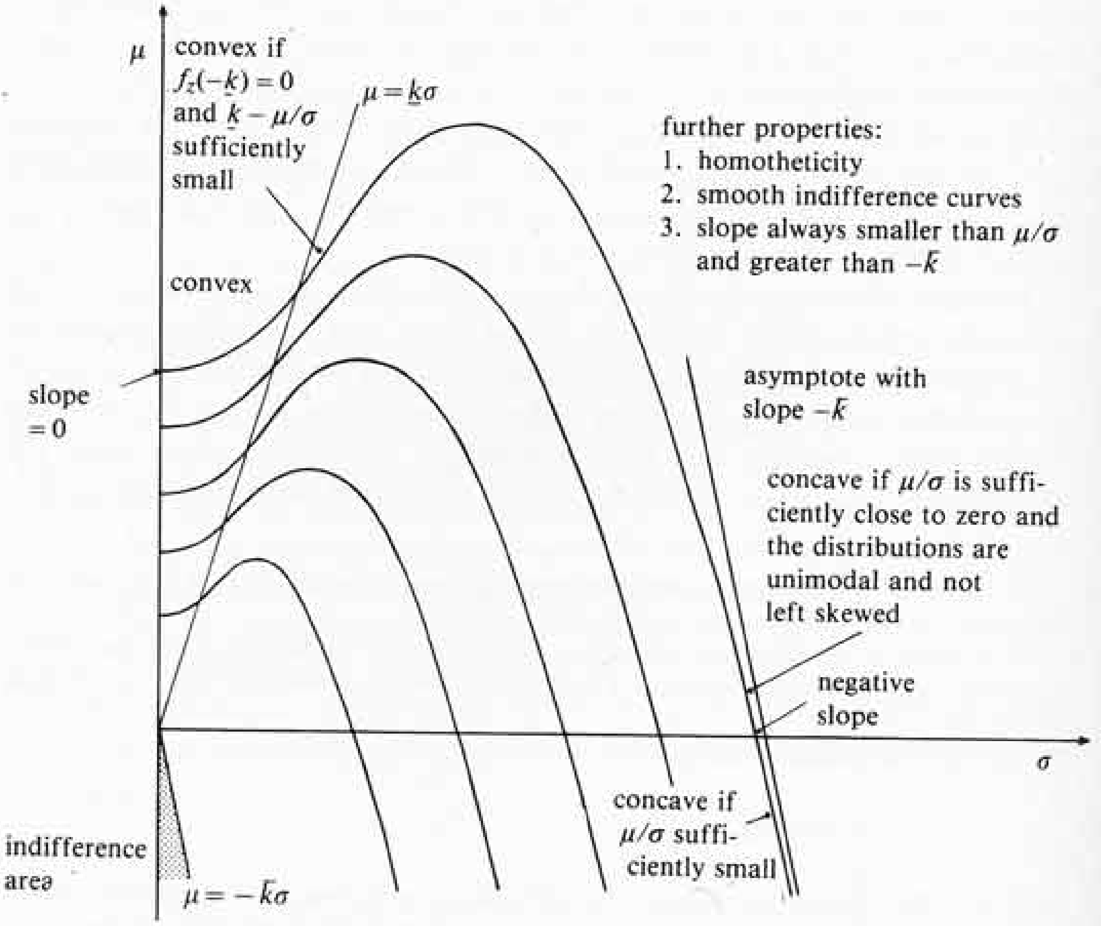


*Figure* 10

More precise information on the indifference-curve slope when $\mu/\sigma$ is in the range $-\bar{k} < \mu/\sigma \leq \underline{k}$ can be gained by inspection of (5). According to this expression, the slope is the negative of a weighted average of the possible variates of $Z$ where the weights are $f_z(z)U'(z+\mu/\sigma)/E[U'(Z+\mu/\sigma)]$. Obviously zero weights are attached to all $z > \bar{k}$, since $f_z = 0$, and to all $z < -\mu/\sigma$, since $U'(v) = 0$ for $v < 0$. Hence $-\mu/\sigma \leq E(ZU')/E(U') \leq \bar{k}$ and thus $-\bar{k} \leq d\mu/d\sigma|_{U(\mu,\sigma)} \leq \mu/\sigma$. Although this information about the indifference-curve slope is rather limited, it confirms the fact that, with $-\bar{k} < \mu/\sigma < 0$, there is a range where the slope is negative.

To obtain further information equation (5) is written in the explicit form

$$\text{(6)} \qquad \left.\frac{d\mu}{d\sigma}\right|_{U(\mu,\sigma)} = -\frac{\int_{-\mu/\sigma}^{\infty} z f_z(z)(1-\varepsilon)\left(\frac{\mu}{\sigma}+z\right)^{-\varepsilon} dz}{\int_{-\mu/\sigma}^{\infty} f_z(z)(1-\varepsilon)\left(\frac{\mu}{\sigma}+z\right)^{-\varepsilon} dz}.$$

Since[10] $\lim_{z \to -\mu/\sigma+} (z+\mu/\sigma)^{-\varepsilon} = \infty$ it is tempting to conjecture that the weight for $z = -\mu/\sigma$ is dominating all others so that $d\mu/d\sigma|_{U(\mu,\sigma)} = \mu/\sigma$ if $f_z(-\mu/\sigma) > 0$. In the range $-\bar{k} < \mu/\sigma < \underline{k}$, the indifference-curve system would then be a set of rays through the origin. Appendix 2 shows this conjecture to be wrong. (Substitute[11] $A \equiv d\mu/d\sigma|_{U(\mu,\sigma)}$, $\mu/\sigma \equiv y$, $z \equiv w$, $\varepsilon \equiv \Theta$.) As long as the assumption $0 < \varepsilon < 1$ that underlies this section is maintained we have

$$\text{(7)} \qquad \left.\frac{d\mu}{d\sigma}\right|_{U(\mu,\sigma)} < \frac{\mu}{\sigma}, \quad \text{if} \quad \frac{\mu}{\sigma} > -\bar{k} \quad \text{and} \quad \infty > f_z\left(-\frac{\mu}{\sigma}\right) > 0.$$

This result will hold *a fortiori* if $f_z(-\mu/\sigma) = 0$, perhaps because the considered type of distribution is multimodal or because[12] $\mu/\sigma > \underline{k}$, the situation analyzed in section A.

It is clear that the lower boundary $-\bar{k}$ of the slope will never be reached when variates of $Z$ in the range $-\mu/\sigma \leq z < \bar{k}$ occur with strictly positive probability. If, however, $\mu/\sigma \to -\bar{k}$ then the range of $z$ values used for calculating $E(ZU')$ becomes even narrower. So we must

---

[10] To distinguish limits from above and from below '+' or '−' are placed after the variable denoting the limit.

[11] In parts of the appendices different symbols are used since the mathematical problems treated are considered in various contexts.

[12] This was shown in equations (A 49)–(A 51).

generally conclude that[13]

$$(8) \qquad \left.\frac{d\mu}{d\sigma}\right|_{U(\mu,\sigma)} > -\bar{k}, \quad \lim_{\mu/\sigma \to -\bar{k}} \left.\frac{d\mu}{d\sigma}\right|_{U(\mu,\sigma)} = -\bar{k}.$$

Hence, for $\mu/\sigma$ sufficiently small, the indifference curves are nearly parallel to the lower boundary $\mu = -\bar{k}\sigma$ of the range where indifference curves exist.

In addition to the information about the slope of the indifference curves, information about the curvature may be of interest. From Tobin's proof presented under point (2) in section II D 2.3 we know that, in the range $\mu/\sigma > \underline{k}$, the indifference curves are strictly convex. To find out about the curvature in the range $-\bar{k} \leq \mu/\sigma \leq \underline{k}$ we differentiate (5) with respect to $\mu/\sigma$. The result of this calculation, carried out in appendix 3, is:

$$(9) \qquad \frac{d\left.\dfrac{d\mu}{d\sigma}\right|_{U(\mu,\sigma)}}{d\dfrac{\mu}{\sigma}} = \frac{\varepsilon(1-\varepsilon)}{\beta} \int_{-\mu/\sigma}^{\infty} \left[f_z\left(-\frac{\mu}{\sigma}\right) - (1-\Gamma) f_z(z)\right] \left(\frac{\mu}{\sigma} + z\right)^{-(1+\varepsilon)} dz \left(\frac{\mu}{\sigma} - \left.\frac{d\mu}{d\sigma}\right|_{U(\mu,\sigma)}\right),$$

$$1 > \Gamma > 0, \quad \text{if} \quad f_z\left(-\frac{\mu}{\sigma}\right) = 0,$$

$$\Gamma = 0, \quad \text{if} \quad f_z\left(-\frac{\mu}{\sigma}\right) > 0,$$

$$\infty > \beta > 0.$$

Since $\beta$, $\varepsilon(1-\varepsilon)$, and, because of (7), the terms in brackets behind the integral are strictly positive and finite, so we only have to consider the integral itself.

As shown in appendix 4, it is finite if, as assumed, $0 < \varepsilon < 1$. (Substitute the integral for $A$ and set $\mu/\sigma \equiv y$, $z \equiv w$, $f_z(-\mu/\sigma) - (1-\Gamma) f_z(z) \equiv$

---

[13] In the case of discrete probability distributions rather than continuous ones that can be described by density functions, the slope may take on the value $-\bar{k}$ even for $\mu/\sigma > -\bar{k}$. The sufficient condition for this case is that the variate $z = -\bar{k}$ obtains with positive probability and that further variates in the range $-\mu/\sigma \leq z < \bar{k}$ are impossible. For a binary distribution, this means that in the *whole* range below the line $\mu = \underline{k}\sigma$ the indifference curves have a slope of size $-\underline{k}$. The discrete distributions and the corresponding indifference-curve systems can be approximated by the use of continuous distributions as closely as we wish.

$f_w(w)$, $(1+\varepsilon)\equiv\Theta$.) Since the utility curve is kinked at $v=0$ the suspicion could arise that the indifference curves are also kinked somewhere, at least on the border line $\mu=\underline{k}\sigma$. The finiteness of the integral allays this suspicion. The indifference curves are smooth everywhere.

Whether the indifference curves are convex, concave, or linear is determined by the sign of the integral in (9). In the case $\mu/\sigma>\underline{k}$ it is negative because of $f_z(-\mu/\sigma)=0$ and $\Gamma<1$; this restates the convexity proved by Tobin. It is worth noting that the sign of the integral stays negative even in the case $\underline{k}-\mu/\sigma>0$, provided that this difference is sufficiently small and provided that the density function has the property $f_z(-\underline{k}+)=f_z(-\underline{k}-)$. Thus the indifference curves stay convex in the neighborhood of the line $\mu=\underline{k}\sigma$ even for $\mu/\sigma<\underline{k}$. If, however, the density function is 'truncated' so that $f_z(-\underline{k}+)>0$ while $f_z(-\underline{k}-)=0$, then immediately below this line there will be a concave segment provided $f_z(-\underline{k}+)$ is sufficiently large.

For a unimodal distribution class the integral is strictly positive if the mode ($M$) is zero or negative for, in this case, $f_z(-M/\sigma)-f_z(z)>0$ $\forall z>-M/\sigma$. Hence, for such a class, the indifference curves are definitely concave if $M/\sigma$ is negative or sufficiently close to zero. In the case of a right skewed or a symmetrical distribution where $M\leq\mu$, this also means that there exists some $x=\text{const.}>0$ such that concavity is ensured for all $\mu/\sigma<x$. In the case of multimodal distributions, the indifference curves may consist of various convex and concave segments. In this case, concavity is only ensured when $\mu/\sigma$ is small enough so that the highest mode is close enough to zero.

The simplest version of an indifference curve system that can normally be expected is shown in Figure 10. The properties that have been derived are labelled.

Up to now, only distributions that are bounded to the left have been considered. This does not seem unrealistic. In many practical problems even the gross distribution of wealth in the sense of a balance sheet item appears to have this property since the maximum loss is often limited to capital ventured in a particular enterprise rather than to the decision maker's personal wealth. We should think here, for example, of forms of speculation that tie up capital, of share holding, or of the participation in other limited-liability enterprises. On the other hand, there are a number of decision problems like speculation by selling short or liability insurance, where wealth distributions that disperse very widely to the left have to be evaluated. These problems legitimate the consideration of the limiting case of distributions that are unbounded to the left. In addition, of course, the normal distribution creates some interest in this case, although it must be admitted that there are hardly any problems where the normal distribution approximates the left tails of the appropriate distributions particularly well.

For $\underline{k} = \infty$, the area of indifference curves where all variates of $V$ are positive no longer exists. For any $\sigma > 0$ the Bloos rule affects the indifference-curve slope. Clearly this aspect does not change those particular conclusions, derived above for $\mu/\sigma < \underline{k}$, that were not confined to the neighborhood of the $\mu = \underline{k}\sigma$ line. Thus, the property of homotheticity will continue to hold, and the indifference curves will be negatively sloped and concave for $\mu/\sigma$ sufficiently small. The question remains, however, of how the indifference curves are shaped in the neighborhood of the ordinate. Are they still approaching the ordinate perpendicularly, and if so, will there still be some range of convex indifference curves in the neighborhood of the ordinate where the decision maker behaves as a risk averter?

The first part of this question can easily be answered. Since, in the present case $0 < \varepsilon < 1$, numerator and denominator of (5) are finite[14], the discontinuous region of $U'(\upsilon)$ at $\upsilon = 0$ can be approximated by a continuous marginal utility function as closely as we wish. In the limit as $\sigma \to 0$, for a continuous marginal utility function, the numerator takes on the value $E(Z)U'(\mu) = 0$ and the denominator the value $U'(\mu)$[15]; moreover the weight of some given range of approximation around $\upsilon = 0$ vanishes[16]. Thus, as before, the indifference curves are horizontal at the ordinate:

$$(10) \qquad \lim_{\sigma \to 0} \left. \frac{d\mu}{d\sigma} \right|_{U(\mu,\sigma)} = 0.$$

---

[14] This followed, e.g., from the calculation of equation (6) in appendix 2.

[15] Cf. Schneeweiss (1967a, pp. 128 f.).

[16] That removing the discontinuity has a negligible influence can be shown by using expression $A$ in appendix 2. First, within equation (5) from the text above, the substitutions $y \equiv \mu/\sigma$, $z \equiv w$, $\varepsilon \equiv \Theta$ are carried out. Then, the shape of the marginal utility curve is made continuous in the interval from $-y$ through $-y+\Delta$ by the use of a function $\chi(w)$ where

$$\int_{-y}^{-y+\Delta} f_w(w)\chi(x+w)\,dw = \int_{-y}^{-y+\Delta} f_w(w)(x+w)^{-\Theta}\,dw$$

is assumed, so that $\gamma$ in equation (9) of appendix 2 remains unchanged. This modification changes the value of $\lim_{x\to y+} \alpha$ by a finite amount. If before the modification $\Delta$ was chosen so as to ensure that $\gamma$ is sufficiently close to unity, then, independently of this modification, equations (11) and (3) of the appendix imply $A \approx \lim_{x\to y+} \beta$ where each desired degree of approximation can be reached. But even if a very high degree of approximation is not desired, the subsequent taking of the limit $y \to \infty (\sigma \to 0)$ implies that $\lim_{x\to y+} \gamma$ approaches unity and hence the question of whether or not the marginal utility curve is continuous in the range from $-y$ through $-y+\Delta$ turns out to be irrelevant as long as $0 < \Theta < 1$. If $\Theta \geq 1$ then, because of equation (21) from the appendix, we always have $\lim_{x\to y+} \gamma = 0$ so that the above reasoning is no longer valid.

The second question can be answered by reference to expression (9). Obviously convexity prevails, despite the Bloos rule in operation, if, in the limit as $\mu/\sigma \to \infty$, the integral $(\int)$ in (9) approaches a strictly negative value. To find out whether this is the case, some formal analysis seems necessary.

Assume that the prevailing linear distribution class has the property

$$f_z(z) > 0, \quad f_z'(z) \geq 0, \quad \text{for} \quad z \leq -\varrho, \tag{11}$$

where $\varrho$ is some constant that is chosen sufficiently large, and let $\mu/\sigma = \varrho + x + y$, $0 < x < \infty$, $0 < y < \infty$. Moreover, write the integral in (9) in the form

$$\begin{aligned} \int = & \int_{-\varrho-x-y}^{-\varrho} [f_z(-\varrho-x-y) - f_z(z)](\varrho+x+y+z)^{-(1+\varepsilon)}\,dz \\ & + f_z(-\varrho-x-y) \int_{-\varrho}^{\infty} (\varrho+x+y+z)^{-(1+\varepsilon)}\,dz \\ & - \int_{-\varrho}^{\infty} f_z(z)(\varrho+x+y+z)^{-(1+\varepsilon)}\,dz. \end{aligned} \tag{12}$$

Then it is possible to derive a sufficient condition for the sign being negative in the limit as $\mu/\sigma \to \infty$. Obviously, by construction, it holds that $\int_{-\varrho-x-y}^{-\varrho} \dots dz \leq 0$. Hence $\int$ is smaller than or equal to the sum of the other two items on the right-hand side of (12).

Consider now the inequality

$$\begin{aligned} & \left(\frac{x+y}{x}\right)^{-(1+\varepsilon)} \int_{-\varrho}^{\infty} f_z(z)(\varrho+x+z)^{-(1+\varepsilon)}\,dz \\ & < \int_{-\varrho}^{\infty} f_z(z)(\varrho+x+y+z)^{-(1+\varepsilon)}\,dz \end{aligned} \tag{13}$$

which follows from the facts that

$$\frac{x+y}{x}(\varrho+x+z) = \varrho+x+y+z, \quad \text{if} \quad z = -\varrho, \tag{14}$$

$$\frac{x+y}{x}(\varrho+x+z) > \varrho+x+y+z, \quad \text{if} \quad z > -\varrho, \tag{15}$$

and that $(.)^{-(1+\varepsilon)}$ is a strictly decreasing function. Utilizing (13), we clearly have

$$\begin{aligned} \int < & f_z(-\varrho-x-y) \int_{-\varrho}^{\infty} (\varrho+x+y+z)^{-(1+\varepsilon)}\,dz \\ & - \left(\frac{x+y}{x}\right)^{-(1+\varepsilon)} \int_{-\varrho}^{\infty} f_z(z)(\varrho+x+z)^{-(1+\varepsilon)}\,dz. \end{aligned} \tag{16}$$

Thus $\lim_{\mu/\sigma \to \infty} \int < 0$ if the right-hand side of (16) becomes negative as $y \to \infty$. This in turn is the case, if the first item on the right-hand side vanishes 'faster' than the absolute value of the second. Since $\frac{\partial}{\partial y} \int_{-\varrho}^{\infty} (\varrho + x + y + z)^{-(1+\varepsilon)} \, dz < 0$ the second term on the right-hand side of (16) will definitely dominate the first if

$$\lim_{y \to \infty} \frac{f_z(-\varrho - x - y)}{(x+y)^{-(1+\varepsilon)}} = 0. \tag{17}$$

Equation (17) therefore is a sufficient condition for the indifference curves being convex in the neighborhood of the ordinate.

As an example, consider the normal distribution where $f_z(z) = (1/\sqrt{2\pi}) e^{-z^2/2}$. Here, with $z \to -\infty$, the density vanishes faster than $e^{-|z|/2}$ or, equivalenty, with $z = -(\varrho + x + y)$ and $y \to \infty$ faster than $e^{-|-\varrho - x - y|/2} = e^{-(\varrho + x)/2} e^{-y/2}$. Since

$$e^{-y/2}/(x+y)^{-(1+\varepsilon)} = e^{-y/2}/e^{-(1+\varepsilon)\ln(x+y)} = e^{(1+\varepsilon)\ln(x+y) - y/2}$$

this implies that the normal distribution meets condition (17), provided that $\lim_{y \to \infty} [(1+\varepsilon)\ln(x+y) - y/2] = -\infty$. Obviously this is the case. Hence, at least for the normal distribution and all distributions whose densities on the left-hand side converge faster, the indifference curves are convex if, given $\mu$, $\sigma$ is chosen sufficiently small.

As a final problem in the analysis of distributions that extend over the negative half of the wealth axis the role of the risk aversion parameter $\varepsilon$ should be briefly considered. Here the result

$$\left. \frac{d \left. \frac{d\mu}{d\sigma} \right|_{U(\mu,\sigma)}}{d\varepsilon} \right|_{\frac{\sigma}{\mu}} > 0 \tag{18}$$

that was previously achieved with (A 48) still prevails, since the reasoning of appendix 1 is completely unchanged.

Thus we may summarize as follows. Although the subjective preferences of the decision maker exhibit risk aversion, albeit moderate because of $\varepsilon < 1$, the BLOOS rule produces risk loving behavior provided the gross distribution extends widely enough over the negative half of wealth axis.

In the case of a linear distribution class bounded to the left at $\mu - \underline{k}\sigma$, $\underline{k} < \infty$, the indifference-curve system in the $(\mu, \sigma)$ diagram has the usual properties in the range $\mu/\sigma > \underline{k}$, but, in the range $\mu/\sigma < \underline{k}$, the indifference curves are negatively sloped and concave for $\mu/\sigma$ sufficiently small. With $\bar{k}$ as the upper boundary of the standardized random variable characterizing the distribution class, in the limit as $\mu/\sigma$ approaches $-\bar{k}$, the indifference-curve slope also approaches this value.

In the case of a linear distribution class unbounded to the left, despite the Bloos rule in operation, the indifference curves have the usual properties in the neighborhood of the ordinate provided that, for $z \rightarrow -\infty$, the density converges at least as fast as that of a normal distribution. For sufficiently small values of $\mu/\sigma$ the question of a boundedness to the left is irrelevant for the shape of the indifference curves. As with all distributions unbounded to the right, with the normal distribution the indifference curves approach vertical asymptotes for $\mu/\sigma$ sufficiently small.

At any point in the $(\mu, \sigma)$ diagram the indifference-curve slope is a rising function of the measure of relative risk aversion $\varepsilon$.

The indifference-curve system is homothetic.

### 1.3. *Critique of the Subjectivist Foundation of Risk Preference*

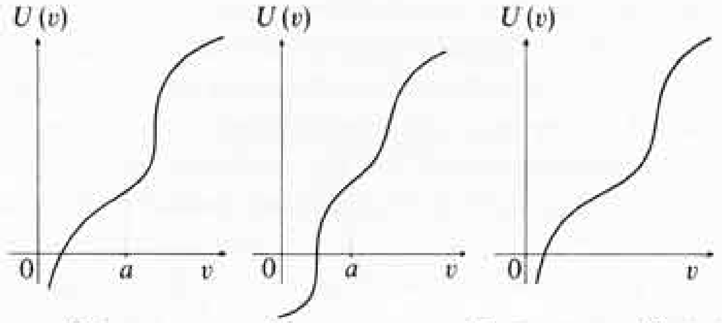
The explanation of risk loving behavior as given by the Bloos rule is at variance with traditional explanations of this behavior. It does not have very much in common with a subjective inclination towards risk except for the fact the utility function has to be bounded for $v \rightarrow 0$. Figure 11 compares the utility curve (2) to the classical curves[17] suggested by Törnqvist (1945), Friedman and Savage (1948), and Markowitz (1952b).

A common feature of the classical proposals is that the convex segments in the medium ranges of the curves are explained by the empirical observation that, despite negative expected net gains, people participate in gambling. Markowitz and Törnqvist place the convex segment to the right of the initial wealth $a$ since the range of gambling seems to be there. This assumption implies that there is no given utility-of-wealth curve but that the positions of the curves are dependent on initial wealth. Friedman and Savage argue that the convex segment should be at medium levels of wealth since comparatively poor people seem to be particularly interested in gambling[18].

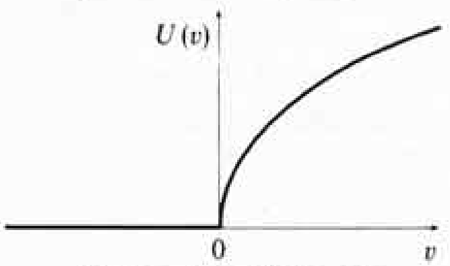
---

[17] Friedman and Savage call the argument of their utility curve 'income'. Their reasoning, however, describes a utility-of-wealth function better. Törnqvist and Markowitz assume that the position of the utility curve depends on the decision maker's initial wealth ($a$).

[18] The authors unanimously explain the concave curve segments to the left of the convex ranges with the preference for buying insurance. The right-hand concave segment is explained by Friedman and Savage with the argument that people seem to have a preference for diversifying prizes in gambling. From a more formal point of view Markowitz and Törnqvist, however, argue that the utility function is bounded from above to avoid the St. Petersburg Paradox. With a similar argument Markowitz finally legitimates a lower boundary to utility which produces a convex segment.



Törnqvist    Markowitz    Friedman and Savage



Bloos rule and Weber's law

*Figure* 11

The classical arguments for a convex segment in the utility function are empirically doubtful, inconsistent, and imply irrational behavior under risk. If Friedman and Savage are right, then the intensity of insurance demand must be very low for people from the upper middle class, and no one from this class would want to play roulette[19]. These implausible implications could be removed by adopting the Törnqvist-Markowitz hypothesis that the total curve shifts with an increase in wealth. However, as we know from the discussion of the Weak Relativity Axiom, this solution demands the high price of violating the Axiom of Ordering[20].

These problems suggest that the idea as such of deriving the shape of the utility curve from gambling behavior is misleading. The shortcomings of this type of reasoning are very obvious in the light of some peculiarities of gambling that are hardly compatible with the von Neumann-Morgenstern axioms.

- First, it is necessary to mention the fun of observing complicated game procedures which, as we know, is incompatible with the Axiom

---

[19] This comment also applies to Hakansson (1970b), Masson (1972), and Appelbaum and Katz (1981) who derive the Friedman-Savage utility function from an intertemporal optimization approach with capital-market imperfections. For an extensive criticism of the Friedman-Savage utility function see Bailey, Olson, and Wonnacott (1980).

[20] Cf. in section A 2.1 the discussion centering around equations (A 36) and (A 37).

of Ordering[21]. This fun may explain why people participate in games of chance, but it does not imply anything for the shape of the utility function for serious economic decision making.

- Moreover there is a good case to be made for ALLAIS's (1952, p. 132) explanation of gambling, namely, that the stake is below a subjective threshold while the prizes are beyond it. Again this argument has no bearing on the kinds of economic decision making we are considering in this book. Thresholds do not seem to be important in insurance demand, portfolio choice, or speculation.
- A related argument is that people are inclined to overestimate small, but underestimate large, probabilities. This argument may also explain gambling since the probabilities of winning are usually very small. At any rate, YAARI (1965) felt that an explanation of this sort was needed since he was unable to detect the convex segment in the utility function in his experimental work on risk preferences.
- Finally, doubts must in principle be raised about applying the von Neumann-Morgenstern theory of rational behavior under risk. The attempt by gamblers to outwit probability theory by their 'crystal ball' strategies is surely a good example of irrational behavior. It seems there is a good deal of wisdom in what HICKS (1962, p. 793) says[22] when comparing gambling with portfolio decisions: 'They [the portfolio decisions] are work; gambling is relaxation. To expect consistency in gambling is futile, for gambling is a rest from consistency.'

Rather than trying to explain gambling behavior, it would be better to try explaining types of risk loving behavior that have nothing to do with the fun of gambling or thresholds in perception. Such behavior is to be observed among people who are clever enough to discover what their optimal decisions are.

Why is it that most people obviously have such a low preference for automobile liability insurance that governments had to make this type of insurance compulsory? Why is the entrepreneur, who is up to his ears in debt, willing to risk everything on one more attempt? Why do shipowners build their tankers like tin cans that break open at the slightest impact and spew their oil into the sea, causing losses far greater than the value of the tanker and its contents put together? In all such cases, there

---

[21] If the fun of gambling is independent of stakes and prizes and is merely added to expected utility then MARKOWITZ (1952b, pp. 157 f.) may be right in saying that a concave utility function implies that '... when millionaires play together, they play for pennies'. If however, the completely implausible assumption of independent utilities is removed then it may well be possible that millionaires play with large stakes, even though in serious and less pleasant economic decision problems their behavior exhibits risk aversion.

[22] Cf. also HICKS (1931, p. 181).

is a particular lack of concern for very large negative variates of the wealth distribution. The Bloos rule, as reflected in the derived utility function, offers an explanation.

## 2. *The Complete Indifference-Curve System in the Case of Strong Risk Aversion* ($\varepsilon \geq 1$)*: The Implicit Lexicographic Ordering*

In contrast to the case $\varepsilon < 1$, it is now impossible to derive a complete von Neumann-Morgenstern utility function for gross wealth similar to (2), for $\varepsilon \geq 1$ means that the Weber functions (A 34) are unbounded at[23] $\upsilon = 0$: $\lim_{\upsilon \to 0} U''(\upsilon) = -\infty$. Nevertheless the Bloos rule remains valid. It is a matter of indifference whether a person loses only his wealth or whether in addition, he is burdened with a debt that he can never repay. Either situation is a complete disaster.

We must thus conclude that at $\upsilon = \tilde{\upsilon} = 0$ there is a *lexicographic* critical wealth level so that a maximization of the survival probability becomes the predominant aim:

$$\text{(19)} \qquad \max W(\upsilon > 0).$$

The fact that a combination of Weber's law and the Bloos rule renders *possible* a lexicographic level of wealth just where $\tilde{\upsilon} = 0$, is compatible with the general discussion of the theory of lexicograhic preferences given in chapter II B. In section 1.2 of that chapter we found that a lexicographic critical wealth level, if it exists, is situated at $\tilde{\upsilon} = 0$.

Given the information (19), for a linear distribution class it is possible to construct pseudo indifference curves in the $(\mu, \sigma)$ diagram. Since the geometrical locus of points with equal survival probability is defined by the condition[24]

$$\text{(20)} \qquad \frac{\mu - \tilde{\upsilon}}{\sigma} = \text{const.},$$

the pseudo indifference curves are rays through the origin; this is illustrated in Figure 12.

However, the total area in the $(\mu, \sigma)$ diagram is not filled with pseudo indifference curves, for the lower ($\underline{k}$) and upper ($\bar{k}$) boundaries of the

---

[23] This property implies a constraint on the range where the Archimedes Axiom is valid. The problem is taken up in the following section C 2.

[24] Cf. equations (II B 5) and (II B 6).

standardized distribution $Z = (V - \mu)/\sigma$ appear on the scene. This, too, is shown by Figure 12.

From below, the area of pseudo indifference curves is bounded by the ray through the origin $\mu = -\bar{k}\sigma$ below which there is an indifference area[25]. The curve is of the same kind as that depicted in Figure 10 and hence we do not have to elaborate upon it.

More important is the upper boundary $\mu = \underline{k}\sigma$, above which, in the case of wealth distributions bounded to the left, there is the range of substitutive indifference curves well known from Figure 6. If a choice has to be made between distributions from this range, then of course the predominant aim of maximizing the survival probability is irrelevant since all of these distributions ensure survival.

In the discussion of Figure 6 the question of how the indifference curves are shaped in the neighborhood of the curve $\mu = \underline{k}\sigma$ was left open. This question will now be considered so that the areas of normal and pseudo indifference curves can be combined without a break. For the case of a bounded utility function ($0 < \varepsilon < 1$), it was shown that (cf. equation (7)) $d\mu/d\sigma|_{U(\mu,\sigma)} < \mu/\sigma$ so that the indifference curves approach the line $\mu = \underline{k}\sigma$ with a slope lower than $\underline{k}$. In the present case $\varepsilon \geq 1$ such a possibility is not excluded. Appendix 2, particularly with equations (25)–(29), shows that, for density functions that at the left continuously approach 0, i.e., that are characterized by $f_z(-\underline{k}+) = f_z(-\underline{k}-) = 0$, there are the following possibilities:

$$(21) \qquad \lim_{\mu/\sigma \to \underline{k}+} \frac{d\mu}{d\sigma}\bigg|_{U(\mu,\sigma)} \begin{cases} = \mu/\sigma & \text{if} \quad \varepsilon \geq 2, \\ < \mu/\sigma & \text{if} \quad \varepsilon < 2. \end{cases}$$

(When using the appendix consider only the calculation of $\lim_{x \to y+} B$, substitute $d\mu/d\sigma|_{U(\mu,\sigma)}$ according to (A 48) for $B$ and set $y \equiv \underline{k}$, $x \equiv \mu/\sigma$, $\Theta \equiv \varepsilon$, and $w \equiv z$.)

Thus, in the case of a comparatively weak risk aversion ($\varepsilon < 2$), it stays true that the indifference curves approach the curve $\mu = \underline{k}\sigma$ at an angle. But if a stronger risk aversion ($\varepsilon \geq 2$) prevails, as is assumed in Figure 12, then the indifference curves lie closely against the curve $\mu = \underline{k}\sigma$. It should be mentioned that this will occur even in the case $1 \leq \varepsilon < 2$, if a 'truncated' density function with $f_z(-\underline{k}-) = 0$ and $f_z(-\underline{k}+) > 0$ prevails. This follows from expressions (2)–(24) in appendix 2.

So far we have only considered the case of distributions bounded to the left. For these distributions the practically relevant part of the $(\mu, \sigma)$ diagram ($\mu/\sigma > -\bar{k}$) is divided into a substitutive and a lexicographic

---

[25] Cf. section B 1.1.

[26] Cf. the remarks on equation (II D 52).

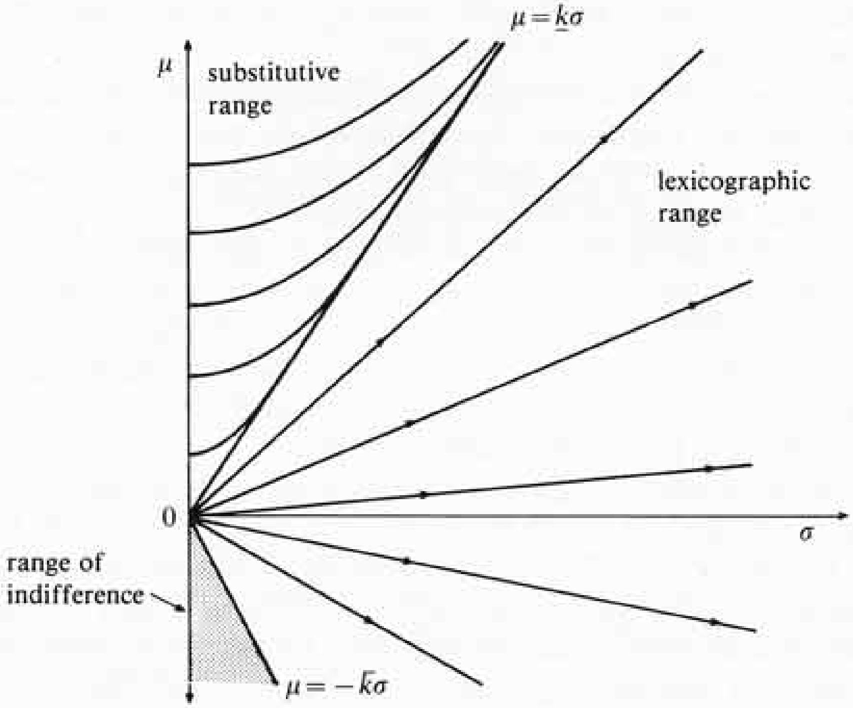


*Figure* 12

area. The difference when the distributions are, like the normal one, unbounded to the left is immediately clear: the substitutive range disappears completely.

For the case of weak risk aversion ($0 < \varepsilon < 1$) we found that with sufficiently small dispersions risk neutrality prevails so that the optimal decision can be based on expected values alone. This rule is clearly violated in the present case of strong risk aversion ($\varepsilon \geq 1$). In the limit as $\sigma \to 0$ even the slightest increase in standard deviation has to be compensated for by an infinite increase in the expected value. This implication appears highly artificial and suggests that the case $\varepsilon \geq 1$ is not a realistic one. On the other hand it should not be forgotten that not only the net, but also the gross (= balance sheet) distributions of wealth are, in practice, often constrained to the left because there are various forms of limited liability in operation. Even the popular normal distribution is, with respect to its left tail, usually not a good approximation of those gross distributions among which economic decision makers have to choose. Thus there might only be a few occasions where unbounded distributions can be observed.

However, regardless of whether or not the $(\mu, \sigma)$ diagram includes a range of substitutive indifference curves, the implications of the lexicographic range as such are not very plausible. The existence of this range implies that people would be willing to pay an insurance premium of almost their initial wealth to get rid of a liability risk that brings about the possibility of negative gross wealth. Obviously this is rarely the case. People are often unwilling to pay premiums that exceed the expected loss by even a moderate amount; these people, at least, do not have lexicographic preferences. This impression will be reinforced by the multiperiod analysis of chapter IV which shows that *only* the case $\varepsilon < 1$ is compatible with the observation that people become more risk averse as they grow older. Thus there is clear evidence against the preference structure depicted in Figure 12, i.e., against a relative risk aversion greater than or equal to unity. But the evidence is only presumptive. Since we cannot ultimately exclude the possibility that $\varepsilon \geq 1$ will hold for at least some people, the analysis should not be confined to the case $0 < \varepsilon < 1$, however attractive this further reduction in the set of possible preference structures might seem.

An open question in the discussion of Figure 12 is how to choose among distributions with an equal survival probability less than unity. Although the expected utility of all these distributions is $-\infty$, people will not generally be indifferent between them. Indeed, it is possible to find dominance rules that allow an ordering to be made. The distributions considered have the property $\upsilon = \mu + z\sigma$, $E(Z) = 0$, $\sigma(Z) = 1$. This implies that a proportional change in $\mu$ and $\sigma$ which does not affect the probability of survival[27] must be an improvement from the viewpoint of the decision maker. The reason is that (2) and

$$\lambda\upsilon = \lambda\mu + z\lambda\sigma, \quad \lambda > 1, \tag{22}$$

ensure that each variate $z$ of the standardized random variable $Z$ is associated with a higher variate $\upsilon^n$ of the net wealth distribution if initially $\upsilon^n > 0$, and is associated with the same variate if initially $\upsilon^n = 0$, i.e., if initially gross wealth was zero or negative ($\upsilon \leq 0$). This improvement, which is immediately plausible, follows from the Axioms of Non-Saturation and Independence. According to these axioms, the decision maker is already better off if a single small interval $\underset{\sim}{z} \leq z \leq \bar{z}$, $\underset{\sim}{z} < \bar{z}$, can be found where the variates $z$ are associated with higher levels of wealth while elsewhere they bring about given levels of wealth. In Figure 12 this result is reflected by the arrows on the pseudo indifference curves. With strict dominance, a movement along such a curve to the right leads to distributions with a higher evaluation.

The most important aspects of the indifference-curve system in the case $\varepsilon \geq 1$ have now been reported. The results can briefly be summarized.

---

[27] Cf. equations (II B 5) and (II B 6) for $\bar{\upsilon} = 0$.

In the case of strong risk aversion ($\varepsilon \geq 1$), Weber's relativity law in connection with the Bloos rule implies that, at $\breve{v} = 0$, there is a lexicographic critical level of wealth. Hence maximizing the probability of survival $W(V > \breve{v})$ is the predominant aim. This aim, however, only has implications for choice if the probability distributions to be evaluated partly extend over the negative half of the wealth axis. If this is not the case, the usual aspects of an evaluation of expected utility remain unaffected.

In the case of linear distribution classes bounded to the left at $\mu - \underline{k}\sigma$, $\underline{k} < \infty$, and to the right at $\mu + \bar{k}\sigma$, $\bar{k} < \infty$, three areas have to be distinguished in the $(\mu, \sigma)$ diagram. An indifference area for $\mu/\sigma \leq -\underline{k}$, an area with rays through the origin as pseudo indifference curves for $-\bar{k} < \mu/\sigma < \underline{k}$, and finally a normal range of substitutive indifference curves for $\mu/\sigma > \underline{k}$. The indifference curves approach the border line between the last two ranges at an angle if $\varepsilon < 2$ and if, on the left side of the distributions, density is continuously declining towards zero. The border line is tangent to the indifference curves if $\varepsilon \geq 2$ and/or the probability distribution is truncated at the left-hand side, i.e., if the density jumps to zero. A pseudo indifference curve ranks above another one if it is situated above it. On a pseudo indifference curve, an increasing distance from the origin means that probability distributions with higher evaluations are reached.

In the case of a linear class of unbounded distributions, for example in the case of the class of normal distributions, the whole $(\mu, \sigma)$ diagram is filled with pseudo indifference curves all centering on the origin.

## Section C

## Arrow's Hypothesis of Increasing Relative and Decreasing Absolute Risk Aversion

Arrow (1965, pp. 28–44; 1970, pp. 90–120) postulates a preference structure that comes, so to speak, half way between the hypotheses of constant absolute and constant relative risk aversion. It implies that an increase in wealth leads to an increase in the intensity of demand for wealth insurance and a decrease in the intensity of demand for insurance of given risk[1].

Crucial to Arrow's argument in favor of the hypothesis of increasing relative risk aversion is his Utility Boundedness Theorem. This theorem requires that, over the positive wealth axis, utility be bounded both from above and below. In deriving his theorem, Arrow (1965, pp. 18–27;

---

[1] Cf. section A 2.3.2.