Factor Content in the 21st Century

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Outline

Technology matrix

- There are n goods and f factors
- Technology matrices come in two forms
- A is an $n \times f$ matrix of direct and indirect input requirements
- In our data, A is a row-stochastic matrix of cost shares
- These numbers are measured without error

Factor conversion matrices, Review

- The technology matrix Θ has dimension 48×3
- The transpose of its pseudo-inverse $(\Theta^{+})^{T}$ is the matrix of Rybczynski effects
- The local factor content of the foreign Rybczynski effects $\Theta_2(\Theta_1^{+})^{T}$ is the factor conversion matrix
- This matrix has dimension $f \times f$.
- The data we observe already have local factor prices in them. Hence they are adjusted for efficiency units

Factor share matrices, Review

- Consider this Leontief technology $A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$
- If local factor prices are $w_1 = (1, 1)^T$, then prices are $p_1 = (2, 3, 4)^T$ and we observe $\Theta_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0.67 & 0.33 \\ 0.75 & 0.25 \end{bmatrix}$

Factor conversion matrices and technology

- Consider a trading partner with the same $A_2 = \begin{bmatrix} 1 & 0.5 \\ 2 & 0.5 \\ 3 & 0.5 \end{bmatrix}$, but its factor prices are $w_2 = (1, 2)$
- Foreign workers are paid more because they are more productive.
- We observe $\Theta_2 = \Theta_1$
- The factor conversion matrix is $\Theta_1^T (\Theta_2^+)^T = I$
- A dollar spent on a worker in country 2 is equivalent to a dollar spent on a worker in country 1.

A better example

- Consider $A_2 = \begin{bmatrix} 1.1 & 0.9 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$, with $w_2 = (1, 1)$
- We observe $\Theta_2 = \begin{bmatrix} 0.55 & 0.45 \\ 0.67 & 0.33 \\ 0.75 & 0.25 \end{bmatrix}$
- The factor conversion matrix is $\Theta_1^T (\Theta_2^+)^T = \begin{bmatrix} 1.0751 & -0.1882 \\ -0.0751 & 1.1882 \end{bmatrix}$
- A dollar spent on a worker in country 2 is equivalent to a \$-0.19 of capital in country 1 and \$1.19 of a worker there

Leontief's factor-specific differences

- $A_1 = A_2 \Pi$
- $\Pi = \begin{bmatrix} \pi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \pi_f \end{bmatrix}$
- These are called factor-specific technical differences

Industry-specific TFP differences

- $A_1 = \Gamma A_2$
- $\Gamma = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \gamma_n \end{bmatrix}$
- If $\gamma_1 = ... = \gamma_n$, then we have Hicks-neutral differences between the countries
- These are considered levels of absolute advantage in international economics, and only the more productive country would produce any homogeneous good.

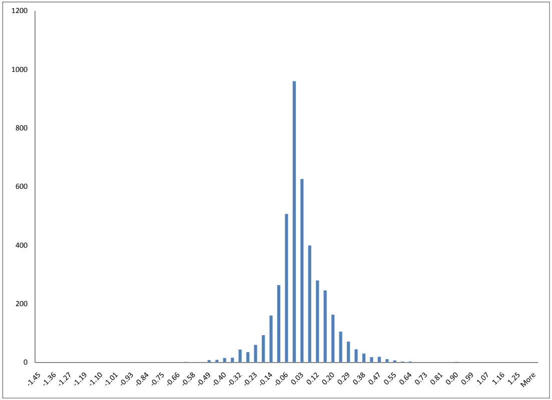
TFP differences and factor-specific differences are not enough

- Let $\gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$ and $\pi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_f \end{bmatrix}$
- Then $\gamma\pi^T$ only has rank 1
- A technology matrix normally has rank *f*
- So this way of thinking about technologies only works exactly when there is *one factor*

Implications of Leontief's factor-specific differences

- If a German worker is twice as efficient as an American worker, then the German wage will be twice that in America
- You run a restaurant in Germany, and your sister runs one in America. You hire one tüchtiger deutscher Arbeiter, and your sister has to hire two lazy American workers
- You both have the same wage bills.
- This is true for every possible industry and for every factor
- *So costs shares are identical for in any sector anywhere in the world*

Factor share differences, USA is reference



Why did Trefler (1993, JPE) find a high correlation between labor productivities and real wages?

Fix your attention on labor. Then Gabaix's (1997) idea is

1. Productivity parameters π_{Li} solve $m_{Li} = \pi_{Li} L_i - s_i \sum_i \pi_{Li} L_i$ for $i = 1, ..., C$ where C is the number of countries
2. When $m_{Li} \approx 0$, $\pi_i = Y_i / L_i$ solves these equations
3. For example, if $i = 1$, we have $Y_1 - s_1 \sum_i Y_i \approx 0$. But this equation just says that country 1's share of world GDP is s_1, a tautology.
4. It is no surprise that countries with high GDP per capita have high real wages!

Leontief's idea is kaput

Implications of TFP difference

- If every industry has an identical capital-labor ratio, then all trade follows Ricardian comparative advantage.
- D. Davis, "Intra-industry Trade: A Heckscher-Ohlin Ricardo Approach", *Journal of International Economics*, 1995
- In a two-country world with only TFP differences, there would be some critical $\tilde{\gamma}$ such that the country in question would export all goods for which $\gamma_i < \tilde{\gamma}$ and import all others
- This is not what we see in the data. All countries export and import all goods

Canonical forms for differences are not born out empirically

- Fix the reference $A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$
- $A_2 = \Gamma A_1 \Pi$ is a general way of modeling these differences
- Since TFP differences do not affect factor shares, we would still have $\Theta_2 = \Theta_1$
- These ideas are strongly refuted by the data

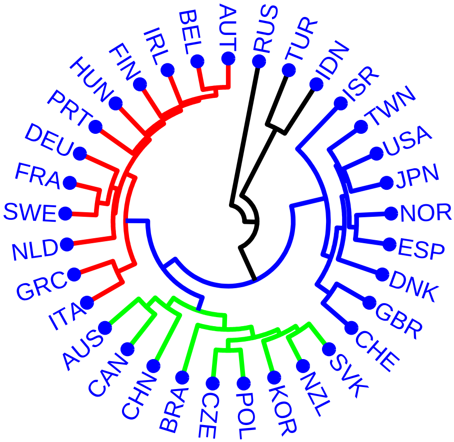
The *p*-norm for a matrix

- A matrix norm is based upon a vector norm
- $\|A\|_p = sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$
- $\|x\|_1 = |x_1| + ... + |x_n|$
- $\|x\|_2 = \sqrt{x_1^2 + ... + x_n^2}$
- $\|x\|_\infty = max\{|x_1|, ..., |x_n|\}$

Matrix 1-norm

- Let $a_j = \sum_i |a_{ij}|$
- $\|A\|_1 = max_j\{a_1, ..., a_f\}$
- *The 1-norm is the maximum of the column sums of the absolute values of the elements.*
- Two countries have different technologies if any factor-specific productivity difference is large.
- Two countries have different technologies if factor shares are very different.

1-norm Clustering



Hierarchical clustering

- The distance between countries i and j is $\|A_i - A_j\|_p$
- I have $33 * 32/2 = 528$ bilateral distances
- Let A and B be two sets. Then $d(A, B) = max\{d(x, y) : x \in A, y \in B\}$.

Details on clustering algorithm

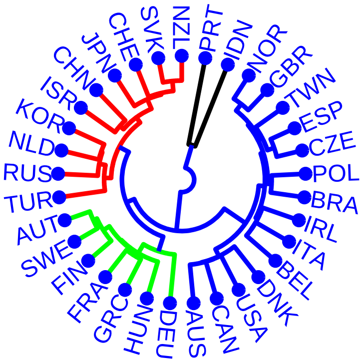
The clustering algorithm proceeds iteratively

1. Identify nearest neighbors
2. Link them using branch heights in the dendrogram
3. When all the initial singletons are linked, the algorithm stops.

Matrix ∞-norm

- Let $a_i = \sum_j |a_{ij}|$
- $\|A\|_\infty = max_i\{a_i, ..., a_n\}$
- *The ∞-norm is the maximum of the row sums of the absolute values of the elements.*
- Two countries have different technologies if any good has a large TFP difference across countries
- Two countries have different technology if any one good has big differences in factor shares. This would occur if there were disparate patterns of indirect business taxes.

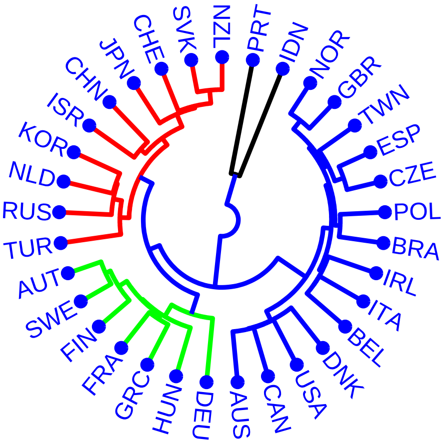
∞-norm Clustering



Matrix 2-norm

- Consider the $f \times f$ matrix $A^T A$.
- Let $\lambda_1, ..., \lambda_f$ be its eigenvalues
- Then $\|A\|_2 = max\{\lambda_1^{1/2}, ..., \lambda_f^{1/2}\}$
- The 2-norm is also called the spectral norm
- The Frobenius norm is $\sqrt{\sum_{i=1}^{f} \lambda_i}$
- Two countries have similar technologies if element-by-element comparisons are not too different

2-norm Clustering



Two inequalities

- Analog of Cauchy-Schwarz inequality is $\|Ax\|_p \leq \|A\|_p \|x\|_p$
- This inequality implies that factor content “dampens” trade volumes
- The triangle inequality is $\|A+B\|_p \leq \|A\|_p + \|B\|_p$
- This inequality implies that factor content measured using an average technology matrix is less than the average of the two local factor contents.

Decomposing a technology matrix

- Consider A the $n \times f$ technology matrix
- We can write $A = USV^T$ where
  - U is an $n \times n$ orthogonal matrix
  - S is an $n \times f$ matrix of diagonal matrix of singular values
  - V is an $f \times f$ orthogonal matrix
- A is a linear mapping from the space of factor prices to that of goods prices

Matrix of singular values

- The singular values of A are the square roots of the eigenvalues of $A^T A$
- The usual case has $n > f$ and A with full rank
- $S = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- It is customary to write $\sigma_1 \geq \ldots \geq \sigma_f \geq 0$

The rotation matrices U and V

- The columns of U are the normalized eigenvectors of AA^T
- The columns of V are the normalized eigenvectors of $A^T A$

Three steps of the linear mapping

- Step 1: $V^T w$ expresses the factor price vector in new coordinates
- Step 2: S expands or contracts these prices
- Step 3: U rotates and also translates this vector into dollar prices in an $n - dimensional$ space

The first singular value decomposition is a good approximation

- Let $n \geq f$
- Let u_i be the $i - th$ column of U and v_i be that of V
- $A = \sum_{i=1}^{f} \sigma_i u_i v_i^T$
- So $\sigma_1 u_1 v_1^T$ may be a good approximation of the technology
- This approximation has rank one, and it corresponds to the idea that factor-specific productivities and sector-specific productivities completely characterizer a technology

Numerical example, part 1

- $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$
- $U = \begin{bmatrix} -0.3231 & 0.8538 & 0.4082 \\ -0.5475 & 0.1832 & -0.8165 \\ -0.7719 & -0.4873 & 0.4082 \end{bmatrix}$
- $S = \begin{bmatrix} 4.0791 & 0 \\ 0 & 0.6005 \\ 0 & 0 \end{bmatrix}$
- $V = \begin{bmatrix} -0.9153 & -0.4027 \\ -0.4027 & 0.91532 \end{bmatrix}$

Numerical example, part 2

- $u_1 \sigma_1 v_1^T = \begin{bmatrix} 1.21 & 0.53 \\ 2.04 & 0.90 \\ 2.88 & 1.27 \end{bmatrix}$
- This approximation has rank 1
- The capital-labor ratios are 2.28
- The TFP's by sector are 0.87^{-1}, 1.47^{-1}, and 2.08^{-1}
- I have explained 87% of the technology this way

Second numerical example

$$\Theta = \begin{bmatrix} 0.5 & 0.5 \\ 0.67 & 0.33 \\ 0.75 & 0.25 \end{bmatrix} \quad u_1 \sigma_1 v_1^T = \begin{bmatrix} 0.5952 & 0.3264 \\ 0.6542 & 0.3587 \\ 0.6820 & 0.3740 \end{bmatrix}$$

Explain 84% of the technology this way

The factor shares do not add to unity.

But the industries with high row sums are the high cost industries.

Singular value decomposition and the Moore-Penrose Inverse

- If $A = USV^T$, then $A^+ = VS^+U^T$
- $S^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 1/\sigma_f & 0 & 0 \end{bmatrix}$

Singular value decomposition and the factor conversion matrix

- Notice that $A_1^T = V_1 S_1^T U_1^T$ and $(A_2^+)^T = U_2 (S_2^+)^T V_2^T$
- Hence $A_1^T (A_2^+)^T = V_1 S_1^T U_1^T U_2 (S_2^+)^T V_2^T$
- $U_1^T U_2$ is orthonormal. So this transformation consists of a rotation, the inverse of an amplification, a rotation, an amplification, and a rotation.
- The ratio of the first singular values of the two technology matrices gives a good summary of relative total factor productivity.

Which country is more productive?

- The ordered singular values for the United States are 4.79, 1.22, and 0.31
- Those for Germany are 4.88, 1.39, 0.73
- Is this *prima facie* evidence that the United States is more productive?

First singular value decomposition

Sector	USA			Germany		
Agriculture	0.323	0.579	0.044	0.387	0.610	-0.045
Mining (energy)	0.290	0.519	0.039	0.666	1.050	-0.077
Mining (non-energy)	0.319	0.571	0.043	0.367	0.578	-0.042
Food products	0.321	0.575	0.043	0.383	0.603	-0.044

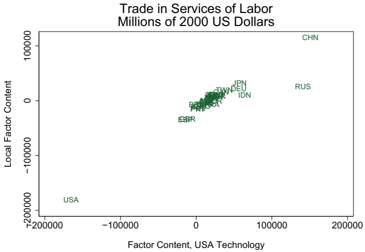
These are the best estimates of one-dimensional descriptions of the two technologies

- In the end, we would like to know how important trade because of differences in endowments is
- This is ultimately an empirical question, but Egger, Fisher, and Marshall have come up with a simple decomposition
- They show that the factor content of trade is explained overwhelmingly by differences in endowments,a s one might expect.

Simple decomposition

- $v_j - s_j v = v_j - s_j \tilde{v} + s_j \tilde{v} - s_j v$
- HOV trade and Ricardian trade:
  $v_j - s_j v = [\tilde{v}_j - s_j \tilde{v}] + s_j [\sum_i (A_j^T - A_i^T) y_i]$
- Empirical implementation:
  $z_i^{0,f} = \beta_0^f + \beta_1^f (\tilde{v}_i^f - s_i \tilde{v}^f) + \beta_2^f s_i \left(\sum_j (A_i(f)^T - A_j(f)^T) y_j \right) + u_i^{0,f}$

Two Measures of Factor Content, Labor Services



SUR Based on USA Factor Content

	Capital (K)		Labor (L)		Social Capital (G)	
	Coef.	Std. Err.	Coef.	Std. Err.	Coef.	Std. Err.
$\hat{\beta}_0$	-3992	2860	612	4073	1483	1263
$\hat{\beta}_1$	0.2730	0.0363	0.3247	0.0671	0.2026	0.0720
$\hat{\beta}_2$	0.0450	0.0220	-0.0595	0.0450	0.0159	0.0158

SUR Based on Local Factor Content

	Capital (K)		Labor (L)		Social Capital (G)	
	Coef.	Std. Err.	Coef.	Std. Err.	Coef.	Std. Err.
$\hat{\beta}_0$	-95	1990	-4859	3931	-362	1315
$\hat{\beta}_1$	0.533	0.024	0.236	0.062	0.458	0.073
$\hat{\beta}_2$	0.009	0.015	-0.037	0.042	0.015	0.016

Empirical content

- We are not measuring 100% of predicted factor trade
- Factor trade because of differences in endowments is much more important than trade because of differences in technology
- Differences in endowments matter about 10 times more than differences in technology
- This fact is reassuring because we *would not* expect Ricardian trade to explain factor content

First Lesson

- It is best to measure the factor content of trade using the local technology. In this way, we measure local resources saved by trade.
- The simplest form of Heckscher-Ohlin Vanek theory does not work
- It is not appropriate to use factor-specific productivity adjustments

Second Lesson

- There are modern data that measure local technologies accurately–measurement error is no longer an issue in this literature
- Countries manage to produce and export almost every good, indicating that local technologies adjust in the long-run to world market conditions
- Local technologies already incorporate local factor prices and thus the proper productivity adjustments for factors
- Differences in total factor productivity by industry are akin to having an absolute advantage in a Ricardian model

Third lesson

- The local factor content of foreign Rybczynski effects is the best way to do cross-country productivity comparisons
- There is little evidence of home bias in consumption, trade costs, or other demand side impediments to trade
- A countries virtual endowment is the vector of resources it would need to produce its actual output if it had the reference county's technology.
- Its virtual endowment the translation of its actual endowment into the factors of the reference country, plus an error term that has no factor content in the country of origin.

Fourth lesson

- The singular value decomposition of a technology matrix uses f one-dimensional matrices to describe a country's technology
- The SVD consist of three steps: (1) a rotation in the space of factors; (2) an amplification or dampening; and (3) a rotation in the space of goods.
- This technique may help us understand overall levels of productivity

Fifth and final lesson

- The factor content of trade is still the best explanation for trade between countries
- Egger, Fisher, and Marshall show that endowment differences account for the overwhelming amount of trade in factor services
- Differences in technologies are of first-order importance, but hey are not arbitrary
- The best way to do economics is to meld theory and empiricism tightly

Edwin Hubble