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Risk Management of Pension Systems from the Perspective of a New Behavioral Life-Cycle Model

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Abstract

Empirically, pay-as-you-go (PAYGO) pension components have a smaller downside risk than funded components. Should reformed pension systems therefore always keep a PAYGO component for optimal risk management? Binswanger (2005) develops a new lexicographic life-cycle model which explains the cross-section of individual risk management much better than existing models with similar degrees of freedom. Here, the new model is applied to the theoretical analysis of optimal pension design. I also provide calibrations suggesting that individuals with a lower income prefer a pure PAYGO system, even in the absence of redistribution. Earners of higher incomes are best off with a mixed system. Overall, income heterogeneity proves to be a crucial issue for optimal pension design.

Key words: Behavioral economics, pension design, risk management.

JEL classification: H55.

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1 Introduction

Many pension reform proposals have been heavily influenced by rate-of-return comparisons between alternative retirement systems. A prominent example is the recent reform plan of President George W. Bush for the US.[1] Decreasing birth rates as well as increasing longevity have shifted the implicit returns of a pay-as-you-go (PAYGO) system downwards. As a result, the popularity of (partially) funded systems has increased substantially. On the other hand, PAYGO systems have had, and will probably continue to have, favorable downside risk properties. Should policy therefore only opt for reform plans that keep a positive PAYGO component? What would be the optimal size of such a component? Apart from the issue of transition, the answer to this question depends mainly on the optimal trade-off between high safety and a high expected level of old age consumption.

Traditionally, economists have used expected utility models such as the constant relative risk aversion (CRRA) or the Epstein-Zin (EZ) model to address optimal risk-return trade-offs. This has provoked many interesting insights. Nevertheless, we should be very cautious when applying these models to policy analysis. The reason is that they do not come close to explain the observed risk-return trade-offs that individuals engage in when deciding about the allocation of their own financial wealth.[2] Empirically, individuals with a lower income do not participate in the stock market. For higher incomes, the empirical expansion path for equity portfolio shares is steeply increasing. In contrast, CRRA and EZ models predict a constant equity share for all income levels. Less restrictive specifications might in principle do a better job in explaining the cross-section of risk-return trade-offs. However, within the class of expected utility models there is no alternative that would be tractable enough to be useful for applied studies. In sum, expected utility models might still be useful in explaining the risk-return trade-offs of a representative agent.[3] But they are less appropriate when income heterogeneity comes into play. Getting back to the risk

[1] See for example Shiller, 2005.

[2] See Haliassos (2002) and Binswanger (2005) for detailed discussions.

[3] The persistence of the equity premium puzzle in economic theory cast some doubt on this, however.

management of pensions, there is thus need for an alternative model if we are not willing to exclude that optimal risk management depends on income. The fact that individual portfolios vary strongly with income levels suggests, indeed, that income heterogeneity is an important issue for pension design.

In Binswanger (2005) I derive a new lexicographic life-cycle model which is based on robust findings from psychological decision research. This model happens to explain the cross-section of individual portfolio choices very well. Moreover, predicted rates of total savings fall well within the range of empirical estimates. This is documented in Figure 1. The dotted lines show the empirical income expansion paths for equity portfolio shares and total saving rates in the upper and lower half of the figure, respectively.[4] The thin solid lines represent the predictions of the EZ model with a relative risk aversion of two and an elasticity of intertemporal substitution of one. The bold solid lines correspond to the lexicographic model. Both, the lexicographic model as well as the EZ model have three parameters that can be chosen freely to match the empirical curves.[5] Evidently, the lexicographic model makes a more efficient use of this threefold degree of freedom. Fortunately, the model is also quite tractable.

In this paper, I use the lexicographic model to explore optimal risk management of retirement incomes. The analysis considers a setting where a pension system is composed of a (defined contribution) PAYGO component, a funded component investing in bonds and a funded component investing in stocks. As a special case it might consist only of a unique such component. The main part of the paper is devoted to the theoretical analysis of the optimal composition of the pension system. The analysis allows for full income heterogeneity. It abstracts, however, from transitional issues and is limited to the consideration of "steady states." Clearly, the analysis of transitions is very important. But a prerequisite for studying potential transition paths is the examination of an appropriate

[4]Equity shares are calculated from the Survey of Consumer Finances 2001. The empirical estimates for saving rates stem from Dynan et al. (2004). See Binswanger (2005) for details.

[5]The parameters of the EZ model are relative risk aversion, the elasticity of intertemporal substitution and the discount factor. The lexicographic model will be discussed in detail below.

"target" system. The analysis here explores the desirability of such target steady states envisaged by any particular reform plan.

A key result of the analysis is that low-income earners are best off with a pure PAYGO system, as long as this system exhibits the lowest downside risk. Earners of higher incomes prefer either a mixed system or a purely funded system. The analysis comes up with a very simple and intuitive condition that allows to discriminate between these two possible scenarios. I use US data on bond and stock returns as well as projected Social Security returns to evaluate this condition. Furthermore, I calibrate for a representative scenario the cross-section of optimal contribution rates. Overall, it turns out that income heterogeneity is a crucial issue for optimal pension design.

The aim of this paper is to contribute to the behaviorally oriented literature on pension design from a new angle. The theoretical strand of the existing behavioral literature mainly examines how an adequate pension design can overcome weaknesses in individual behavior. These weaknesses consist of mental accounting (Shefrin and Thaler, 1988), of lacking willpower to save (Shefrin and Thaler, 1988, Thaler, 1994, Thaler and Benartzi, 2001), and, related, of hyperbolic discounting (Laibson, 1996, Laibson et al., 1998). Furthermore, individuals are reluctant to face decisions (Choi et al., 2003). The lexicographic life-cycle model underlying the analysis of this paper should not a priori be understood as capturing a particular variety of erroneous behavior. What makes this model "behavioral" is its aim to picture decision making in a way consistent with robust findings from psychological decision research. As discussed in Binswanger (2005), it captures key elements of bounded rationality as well as a reference-dependent evaluation of choice prospects. However, this reference-dependency should not be understood as a feature of irrationality. As a result, we may assign normative power to the model in order to analyze *optimal* ways of pension risk management.

Important contributions to the literature on risk management of retirement incomes are Abel (2001), Campbell et al. (2001), Feldstein and Ranguelova (2001), Burtless (2003), Diamond and Geanakoplos (2003), Poterba et al. (2003), Poterba (2004) and Shiller (2005).

The rest of the paper is organized as follows. Section 2 briefly discusses the empirical downside risk of alternative pension components. Section 3 introduces the lexicographic preference model. Section 4 considers optimal risk management from a theoretical point of view. Section 5 provides some representative calibrations. Section 6 concludes.

2 Downside Risk of Alternative Pension Components

This section provides some descriptive evidence for the assertion that the (implicit) returns of a PAYGO system exhibit a smaller downside risk than stock and bond returns. The latter account for the returns of a funded system. The most direct way to compare the alternative return distributions is to look at their means and standard deviations. In the absence of any redistributive element, the implicit return of a PAYGO system is determined by the growth rate of aggregate wage income. For the US, from 1929 to 2004, the geometric average of real annual wage income growth has been 3.6 percent. The corresponding standard deviation amounts to 5.5 percent.[6] For long-term government bonds the geometric mean of real returns has been 2.2 percent with a standard deviation of 9.6 percent. For stocks the corresponding figures are 6.8 and 20.8 percent.[7]

Clearly, the differences in variability between the three rates of return are substantial. Yet, the implications for old-age provision are somewhat vague. To obtain a more concise picture, I make use of the above figures calculating cumulative return distributions for a time horizon of 30 years. Concerning financial returns it is usually assumed that their logarithms are normally distributed. As a result, level returns are distributed lognormally. A further common assumption is that the distributions are identical and independent over time. In order to compare the downside risks of stocks and bonds with the implicit returns of a PAYGO system, I extend the above assumptions to aggregate wage growth.[8]

[6]Source: NIPA Table 1.10. Wage income is measured by compensation of employees and deflated by the GDP deflator.

[7]Figures are taken from Burtless (2003).

[8]An alternative way to gauge risks would be to look at the bottom end of historically realized returns. This would yield a misleading picture, however. The reason is that, historically, minimum realized stock

Figure 2 depicts the inverse of the cumulative distribution functions for 30-year returns. In other words, the curves assign each probability on the horizontal axes a corresponding level of return that is undercut with this particular probability. The bold solid line corresponds to historical wage income growth. The dashed line represents bond returns, the dotted line corresponds to stocks. The figure shows that the difference in downside risk between historical wage income growth and financial market returns is huge. For example, the first percentile for wage income growth amounts to plus 40 percent. The corresponding figures for bonds and stocks are minus 65 and minus 49 percent, respectively.[9] Thus, from a historical perspective, the downside risk of a PAYGO system is very favorable. This is true for the entire range of probabilities shown in Figure 2. (Remember that even average aggregate wage income growth has been higher than average bond returns.)

Due to the decline in fertility rates, the historical growth rates of wage income are of somewhat limited relevance for gauging the distribution of future PAYGO returns, however. I therefore recalculate their cumulative distribution function under the assumption that mean returns correspond to the intermediate long-term projections of the Social Security Administration. For productivity growth they amounts to 1.6 percent, for population growth to 0.2 percent.[10] The resulting estimate for mean annual wage income growth is thus 1.8 percent. The variance is again set to 5.5 percent.

The resulting downside risk of such reduced PAYGO returns is represented by the thin solid line in Figure 2. Although the general level of returns is substantially lower, the distribution continues to be favorable at the bottom. For example, there is a 99 percent probability that PAYGO returns will exceed minus 36 percent. (Remember that this contrast with minus 65 and minus 49 percent for stocks and bonds, respectively.)

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returns have been very high over the long run. In particular, *minimum* stock returns have always been higher than minimum bond returns and mostly even higher than maximum bond returns (see for example Burtless, 2003). If this would have also been true ex ante, nobody should have ever invested in bonds. As a result, historical minimum returns do not provide useful information about ex ante risk.

[9]These figures translate into annual returns of minus 3.4 and minus 2.2 percent, respectively. The annual rate for the PAYGO system is 1.1 percent.

[10]Board of Trustees (2005), Table V.B1 and V.B2. Short-run projections are higher.

One might argue that focusing on the very bottom of the return distributions leads to an excessively conservative measurement of downside risks. In this respect, Figure 2 provides a quite intriguing lesson. Suppose that we neglect the lower tails of the distributions and focus on the fifth (or any higher) percentile, instead. From such a perspective, stock returns would first-order dominate bond returns. As a result, one should never invest in bonds when facing a choice between these two assets. Alternatively one would have to conclude that historical bond and stock returns are greatly misleading for gauging their return distributions.

In the theoretical analysis that follows I will take it as given, that the downside risk of a PAYGO system is lower than for financial returns. Otherwise, there would never be a case for a PAYGO system from the perspective of a steady state analysis. I explore critical limits for expected and minimum returns of a PAYGO component such that it is still valuable from the perspective of risk management. Moreover, it is considered under which conditions the optimal pension system is a pure PAYGO system, a mixed or a purely funded system. The answer to these questions might naturally differ between various income groups.

3 Preferences

The analysis draws on Binswanger (2005), assuming that preferences are lexicographic. This means that preferences are constituted by a list of hierarchically ordered "goals" that a decision maker seeks to achieve. I will refer to these goals as so-called priorities. Mathematically, these priorities should simply be understood as functions of the decision variables. A decision alternative is most preferred if it leads to the highest feasible value of the first priority. If there are several choices remaining for which this is true, the decision maker selects the alternative which leads to the highest feasible value of the second priority etc.

To be specific, I consider a two-period retirement saving model where preferences are lexicographic over the list

$$\mathbf{p} = \left\{ I\left(c^{2\min} \geq \alpha c^1\right) \cdot \min\left\{c^1, R\right\}, \; E\widetilde{c}_2\left(c^1 - R\right)^\beta \right\}. \tag{1}$$

There are thus two priorities. c^1 and $\widetilde{c}^2$ denote working age and retirement consumption, respectively. The tilde indicates that retirement consumption is random because it will be the outcome of risky investments. $c^{2\min}$ denotes the minimum realization of old-age consumption. For the specific economic setting that will be considered, it is guaranteed that a strictly positive minimum consumption is feasible.

Let us consider the first part of the first priority. α represents a parameter with $0 < \alpha < \infty$. I denotes the indicator function. Thus $I\left(c^{2\min} \geq \alpha c^1\right)$ takes a value of one if $c^{2\min} \geq \alpha c^1$ and zero otherwise. As discussed above, decision makers seek to achieve the highest possible value of the first (and then the second) priority. As a result, they will never choose a consumption/savings plan for which $c^{2\min}$ falls short of αc_1, if possible. In other words, agents seek to maintain their standard of living. I refer to the expression αc_1 as a habit level of consumption. Clearly, the requirement that this habit level must not be undercut with probability one is extreme and should not be understood literally. The idea is that consumers want to assure their standard of living with a very high probability. The analysis turns out to be particularly tractable if we focus on the limit case where this probability approaches one.

Consider now the second part of the first priority. This expression takes a value of c_1 or R, whichever is smaller. R is a further parameter of the model that represents a normal standard of living.[11] Overall, the first priority implies that decision makers will maximize c_1 subject to $c^{2\min} \geq \alpha c^1$. This maximization is, of course, subject to the decision makers budget constraint. If the solution of this problem leads to a value of $c_1 \geq R$, then the first priority is "satiated," so to speak. As a result, the second priority comes in to play.

The second priority takes the role of a residual goal that determines behavior conditional on satiation of the first priority. E denotes the mathematical expectation operator.

[11] Clearly, standards of living increase over time. The analysis of this paper abstracts from economic growth, however.

β represents the third parameter of the model. It is assumed that $0 < \beta < \infty$. Essentially, the second priority formalizes the notion of "more is better," be it working age or retirement consumption. The level of c_1 is adjusted by R. This assures that c_1 is a strictly normal good at all income levels. β determines the degree of complementarity between active and retirement consumption.

The first priority in (1) specifies a strong aversion do downside risks. The habit level constitutes a reference point that individuals do not want to be undercut. This is the only source of risk aversion in the model, however. Holding fixed a particular minimum consumption level, there is no disutility from a high dispersion of old-age consumption. As a result, the benefits from portfolio diversification are limited to the lower end of the consumption distribution. From a conceptual point of view, the specification of a more general shape of risk aversion is easily possible. For example, one might substitute the certainty equivalence formulation $\ln E\tilde{c}^2$ in the second priority by $E\left[\ln \tilde{c}^2\right]$. There are three reasons why preferences (1) are nevertheless a more appropriate starting point for the analysis of this paper. First, all crucial insights can be developed without referring to a more general notion of risk aversion. The restriction to downside risk aversion makes the analysis much more tractable. Second, the implications of variability aversion on diversification are well understood. The analysis does not have anything to contribute in this respect. Third, downside risk is by far the most important risk an analysis of pension design must be concerned about. The reason is that most if not all pension systems have been introduced to guarantee a certain minimum standard of living and to reduce old-age poverty.

Mathematically, the outcome of maximizing the second priority in (1) will not be affected by a monotonic transformation. Thus, preferences may equivalently be stated as

$$\mathbf{p}' = \left\{ I\left(c^{2\min} \geq \alpha c^1\right) \cdot \min\left\{c^1, R\right\}, \ln\left(E\widetilde{c}_2\right) + \beta \ln\left(c^1 - R\right) \right\}. \tag{2}$$

It should becomes clear from this that β simply represents the inverse of a standard discount factor. It turns out that it will be more convenient to work with preferences (2) instead of (1) when examining optimal investment choices.

4 Optimal Investment Choices

I consider a setting with three risky assets: bonds, called the x-asset, with random gross return $\widetilde{x}$; stocks, called the y-asset, with random gross return $\widetilde{y}$; and the PAYGO "asset", referred to as z-asset, with random implicit return $\widetilde{z}$. Asset returns are characterized by the following properties.

Assumption 1
(i) $\widetilde{x} \in [\underline{x}, \overline{x}]$ *where* $\underline{x} > 0$.
(ii) $\widetilde{y} \in [0, \overline{y}]$.
(iii) $\widetilde{z} \in [\underline{z}, \overline{z}]$ *where* $\underline{z} > \underline{x}$.
(iv) $E\widetilde{y} > E\widetilde{x} > E\widetilde{z}$.
(v) $\Pr[\widetilde{x} = \underline{x}, \widetilde{y} = 0, \widetilde{z} = \underline{z}] > 0$.

According to (i), (ii) and (iii), the z-asset has the lowest downside risk, followed by the x- and y-asset, respectively. Concerning expected returns, the order is reversed by (iv). By (v), I assume that there is a positive probability that all three returns take on their minimum values. This implies that the minimum payout of a portfolio cannot be increased by diversification. This assumption allows to expose the crucial insights of the analysis by keeping the setting simple. The implications of relaxing this assumption are conceptually straightforward.

By assuming $\underline{y} = 0$, I focus on the limit case where stocks do not contribute at all assuring the habit level αc_1. With $\underline{y} > 0$, the habit level αc_1 could possibly be assured exclusively by equity investments. This possibility would render the solution of the Kuhn-Tucker programs presented below rather cumbersome. From an empirical point of view, the case of pure equity portfolios is irrelevant (see Binswanger, 2005). As a result, no essential insights are lost by focusing on $\underline{y} = 0$. The implications of strictly positive values for $\underline{y}$ will be considered when it comes to the calibration of optimal contribution rates in Section 5.

The analysis proceeds by deriving the portfolio a lexicographic decision maker would choose if she could privately invest in the PAYGO or z-asset. The share of income that will be spent on the z-asset corresponds then to the optimal contribution rate of a PAYGO system in a steady state. Consider an individual with an endowment e in the first and zero in the second period. The individual can use its endowment for first period consumption c^1, for investments in the x-asset, denoted by p (for private precautionary investments), for investments in the z-asset, q, and for y-investments, denoted by s (for speculative saving). In the second period the individual has to live from her investment returns. The individual budget constraints are thus given by

$$c^1 + p + q + s = e,$$
$$p\widetilde{x} + q\widetilde{z} + s\widetilde{y} = \widetilde{c}^2$$

for the first and second period, respectively. I assume that no short-sellings are possible.

Assumption 2 $p \geq 0,\ q \geq 0,\ s \geq 0.$

Due to the hierarchical structure of preferences, optimal choices for low and high endowments have to be considered separately. The reason is that satiation of the first priority in (2) can only be achieved if endowments are sufficiently high. I begin with the low-income case. Define $e_R^{\underline{z}} \equiv \frac{\alpha+\underline{z}}{\underline{z}}R$. We have then

Proposition 1 *For $e \leq e_R^{\underline{z}}$ the optimal consumption/savings choice is given by,* $c^{1*} = \frac{\underline{z}}{\alpha+\underline{z}}e \leq R$, $q^* = \frac{\alpha}{\alpha+\underline{z}}e$, $p^* = 0$, $s^* = 0$.

Proof. Suppose that $p > 0$. Consider the reallocation $dp \in [-p, 0)$, $dq = -\frac{1}{\underline{z}}\left[\underline{x} + \alpha\frac{\underline{z}-\underline{x}}{\alpha+\underline{z}}\right]dp > 0$, $dc^1 = -\frac{\underline{z}-\underline{x}}{\alpha+\underline{z}}dp > 0$ which satisfies $dp + dq + dc^1 = 0$. Thus, the reallocation is feasible and doesn't affect s. Use $c^{2\,\min} = \underline{x}p + \underline{z}q$ to show that this reallocation implies $dc^{2\,\min} = \alpha dc^1$. But since $dc^1 > 0$ the first argument of (2) increases as long as $c^1 \leq R$. We conclude from this that $p^* = 0$ if for any $p > 0$ the indicated reallocation does not lead to a value of $c^1 > R$. If $p^* = 0$ then the optimal choice c^{1*}, q^* is obtained by applying Proposition 1 to c^1 and q (instead of p). To see this note that $\widetilde{z}$ fulfils Assumption 1 if interchanged with $\widetilde{x}$. Thus, $c^{1*} = \frac{\underline{z}}{\alpha+\underline{z}}e$, $q^* = \frac{\alpha}{\alpha+\underline{z}}e$ (and $s^* = 0$). Since $e \leq \frac{\alpha+\underline{z}}{\underline{z}}R$ we have $c^{1*} \leq R$, indeed. ■

Proposition 1 establishes that low-income individuals would exclusively invest in the PAYGO asset, if this would be feasible. This asset exhibits the highest minimum return. As a result, the same level of $c^{2\min}$ can be realized by a smaller amount of precautionary savings in comparison to a situation where only the x- and y-asset are available. Ultimately, a higher value of the first priority can be achieved when the PAYGO asset is available.

If $e > e_R^z$, satiation of the first priority is feasible. Thus, it is feasible to achieve $c^1 > R$ conditional on $c^{2\min} \geq \alpha c^1$. As a consequence, individual behavior is determined by the second preference priority for $e > e_R^z$. Specifically, individuals maximize $\ln(E\widetilde{c}_2) + \beta \ln(c^1 - R)$, conditional on keeping the first argument on its maximal value R. Thus, the maximization is subject to the constraints $c^{2\min} \geq \alpha c^1$ and $c^1 \geq R$ (as well as the budget constraint). Note that the marginal value of an increase in c_1 is infinite at R. This assures that $c^1 > R$. Thus, the constraint $c^1 \geq R$ need not be explicitly taken into account. Using $c^1 + p + q + s = e$, $\widetilde{c}^2 = p\widetilde{x} + q\widetilde{z} + s\widetilde{y}$, Assumption 1 and 2, the optimal choice solves the program

$$\max_{p,q,s} \ln\left[pE\widetilde{x} + qE\widetilde{z} + sE\widetilde{y}\right] + \beta \ln\left[e - p - q - s - R\right] + \lambda\left[p\underline{x} + q\underline{z} - \alpha\left(e - p - q - s\right)\right] + \mu s + \nu p + \xi q \tag{3}$$

The Kuhn-Tucker conditions for this program are stated in the Appendix. The optimal solution can be classified according to the regime $E\widetilde{y} - E\widetilde{z} < \frac{\underline{z}}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$, which is covered by Proposition 2 below, and the regime $E\widetilde{y} - E\widetilde{z} > \frac{\underline{z}}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$, examined in Proposition 3. (Remark 2 will deal with $E\widetilde{y} - E\widetilde{z} = \frac{\underline{z}}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$). Within each regime optimal portfolios differ between several regions defined in the β/e-space. For the regime $E\widetilde{y} - E\widetilde{z} < \frac{\underline{z}}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$, there are two relevant regions, delimited by a critical income level denoted by $\overline{e}^z$ (see the Appendix for a definition). This critical income level is a hyperbolic function of β with jump discontinuity at $\overline{\beta}^z$ (see Figure 3). For each of the two regions optimal decisions are as follows.

Proposition 2 *Assume* $e > e_R^{\underline{z}}$ *and*

$$E\widetilde{y} - E\widetilde{z} < \frac{\underline{z}}{\underline{x}} \left(E\widetilde{y} - E\widetilde{x} \right). \tag{4}$$

(i) If $e \leq \overline{e}^{z}$ *or* $\beta \geq \overline{\beta}^{z}$, *then* $c^{1*} = \frac{\underline{z}}{\alpha+\underline{z}} e$, $q^{*} = \frac{\alpha}{\alpha+\underline{z}} e$, $p^{*} = 0$, $s^{*} = 0$.

(ii) If $e > \overline{e}^{z}$ *and* $\beta < \overline{\beta}^{z}$, *then*

$$\begin{aligned} c^{1*} &= \frac{1}{1+\beta} R + \frac{\beta}{1+\beta} \frac{\underline{z} E\widetilde{y}}{\underline{z} E\widetilde{y} + \alpha \left(E\widetilde{y} - E\widetilde{z} \right)} e, \\ q^{*} &= \frac{\alpha}{\underline{z}} \left[\frac{1}{1+\beta} R + \frac{\beta}{1+\beta} \frac{\underline{z} E\widetilde{y}}{\underline{z} E\widetilde{y} + \alpha \left(E\widetilde{y} - E\widetilde{z} \right)} e \right], \\ s^{*} &= \frac{1}{1+\beta} \left[\frac{\left(\alpha + \underline{z} \right) E\widetilde{y} - \alpha \left(1+\beta \right) E\widetilde{z}}{\underline{z} E\widetilde{y} + \alpha \left(E\widetilde{y} - E\widetilde{z} \right)} e - \frac{\alpha + \underline{z}}{\underline{z}} R \right] > 0, \\ p^{*} &= 0. \end{aligned}$$

Proof. See Lemma 3 and 11 in the Appendix. ■

Optimal portfolios are illustrated graphically in Figure 3. The first part of the Proposition shows that for income levels between e_R^{z} and $\bar{e}^{z}$ it is still optimal not to invest in stocks. The reason is that for such incomes, c_1 is still sufficiently close to R. As a result, the marginal value of an increase in c^1 is very high relative to the marginal value of an increase in $E\tilde{c}^2$. Thus, the marginal value of equity investments is low relative to a marginal increase in c^1. As a result, it is optimal to choose $s = 0$. The only reason to save is still to maintain the habit level αc^1. Thus, optimal portfolios coincide with the case where only the first priority is decisive for behavior. The same is true if β is very high. As stated in part (ii), it is only when income sufficiently exceeds e_R^{z} (and β is not too high), that equity investments are positively valued. In this case, the marginal value of increases in c^1 and $E\widetilde{c}^2$ are balanced.

To understand the meaning of condition (4), consider a portfolio with $p > 0$, $q = 0$, $s > 0$. Thus, this portfolio contains no z-investments. Let us examine whether it is possible to do better by substituting the x-asset with the z-asset. Since $\underline{z} > \underline{x}$, less resources are required to maintain a fixed level of $c^{2\min}$. This is a gain resulting from the substitution. The value of this gain increases with $\frac{\underline{z}}{\underline{x}}$. At the other hand, there is a loss as $E\widetilde{c}^2$ will decrease, ceteris paribus. The reason is that $E\widetilde{z} < E\widetilde{x}$. This loss increases with the difference between $E\widetilde{x}$ and $E\widetilde{z}$. However, if $E\widetilde{y}$ is sufficiently high, the loss can be compensated by increasing y-investments. These can be financed by the resources left over when a fixed level of $c^{2\min}$ is maintained by z-investments rather than x-investments. In sum, $E\widetilde{y}$ is sufficiently high to overcompensate the loss in $E\widetilde{c}^2$ if (4) holds. (With respect to the converse, (4) or the analogous condition under strict equality must hold. In the latter case, optimal portfolios are not unique since the gains and losses from substituting the x-asset with the z-asset just balance.) Alternatively, (4) can be understood as indicating the attractiveness of the z-asset. To see this note that the higher $E\tilde{z}$ and the higher $\underline{z}$, the more likely it will hold.

At first sight, it may seem surprising that bond investments are zero also in the case of positive equity investments. To understand this, note that under preferences (2) there are three carriers of value, so to speak. These are c^1, $E\widetilde{c}^2$, and $c^{2\min}$. There are always at most two assets constituting the portfolio that achieves the most efficient combination of these. (There is one exception, given by the case where the strict equality analogue of (4) holds.) Two assumptions are crucial for this result. First, as discussed in Section 3, there are no benefits of diversification under preferences (2), except for minimum consumption. Second, Assumption 1(v) excludes that diversification at the minimum is operational. Relaxing these assumptions would make it beneficial to hold some positive amount of bonds. However, the two most important conclusions of the proposition would remain unchanged. First, low-income earners will be most concerned about the downside risk of a particular investment opportunity. Second, equity investments are only valuable if income is sufficiently high.

I will now proceed with the regime $E\widetilde{y} - E\widetilde{z} > \frac{\underline{z}}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$. The regions comprising isomorphic portfolios are now delimited by some critical income levels denoted by $\overline{e}^x$, e^p, e^q. These are again hyperbolic functions of β with jump discontinuities at $\overline{\beta}^x$, β^p, β^q, respectively (see again the Appendix for definitions). These functions define four alternative regions in the β/e-space (see Figure 4). The statements in the following remark can be checked by direct calculations.

Remark 1 *(i)* $\beta^p < \beta^q$. *(ii) for* $E\widetilde{y} - E\widetilde{z} > \frac{\underline{z}}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$ *we have* $\overline{\beta}^x < \beta^p$. *(iii) Either* $0 < e^q < e^p$ *or* $e^p < 0$.

For each region in the β/e-space optimal choices are given as follows.

Proposition 3 *Assume* $e > e^z_R$ *and*

$$E\widetilde{y} - E\widetilde{z} > \frac{\underline{z}}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right). \tag{5}$$

(a) If $e \le \overline{e}^x$ *or* $\beta \ge \overline{\beta}^x$, *then* $s^* = 0$. *Moreover,*

- *(i) if* $e \le e^q$ *or* $\beta \ge \beta^q$, *then* $c^{1*} = \frac{\underline{z}}{\alpha+\underline{z}}e$, $q^* = \frac{\alpha}{\alpha+\underline{z}}e$, $p^* = 0$.
- *(ii) if* $e \ge e^p$ *and* $\beta < \beta^p$, *then* $c^{1*} = \frac{\underline{x}}{\alpha+\underline{x}}e$, $p^* = \frac{\alpha}{\alpha+\underline{x}}e$, $q^* = 0$.
- *(iii) If* $e^q < e < e^p$ *and* $\beta \le \beta^p$ *or* $e > e^q$ *and* $\beta^p \le \beta < \beta^q$, *then*

$$\begin{aligned}
c^{1*} &= \frac{1}{1+\beta}R + \frac{\beta}{1+\beta}\frac{\underline{z}E\widetilde{x} - \underline{x}E\widetilde{z}}{\left(\alpha+\underline{z}\right)E\widetilde{x} - \left(\alpha+\underline{x}\right)E\widetilde{z}}e,\\
p^* &= \frac{1}{\left(1+\beta\right)\left(\underline{z}-\underline{x}\right)}\left[\frac{\underline{z}\left(\alpha+\underline{z}\right)E\widetilde{x} - \left[\underline{z}\left(\alpha+\underline{x}\right) + \alpha\beta\left(\underline{z}-\underline{x}\right)\right]E\widetilde{z}}{\left(\alpha+\underline{z}\right)E\widetilde{x} - \left(\alpha+\underline{x}\right)E\widetilde{z}}e - \left(\alpha+\underline{z}\right)R\right]\\
q^* &= \frac{1}{\left(1+\beta\right)\left(\underline{z}-\underline{x}\right)}\left[\frac{\left[\alpha\beta\left(\underline{z}-\underline{x}\right) - \underline{x}\left(\alpha+\underline{z}\right)\right]E\widetilde{x} + \underline{x}\left(\alpha+\underline{x}\right)E\widetilde{z}}{\left(\alpha+\underline{z}\right)E\widetilde{x} - \left(\alpha+\underline{x}\right)E\widetilde{z}}e + \left(\alpha+\underline{x}\right)R\right].
\end{aligned}$$

(Note that the region $e^q < e < e^p$, $\beta^p < \beta < \beta^q$ *is empty.)*

(b) If $e > \overline{e}^x$ *and* $\beta < \overline{\beta}^x$, *then*

$$\begin{aligned}
c^{1*} &= \frac{1}{1+\beta}R + \frac{\beta}{1+\beta}\frac{\underline{x}E\widetilde{y}}{\underline{x}E\widetilde{y} + \alpha\left(E\widetilde{y} - E\widetilde{x}\right)}e,\\
p^* &= \frac{\alpha}{\underline{x}}\left[\frac{1}{1+\beta}R + \frac{\beta}{1+\beta}\frac{\underline{x}E\widetilde{y}}{\underline{x}E\widetilde{y} + \alpha\left(E\widetilde{y} - E\widetilde{x}\right)}e\right],\\
s^* &= \frac{1}{1+\beta}\left[\frac{\left(\alpha+\underline{x}\right)E\widetilde{y} - \alpha\left(1+\beta\right)E\widetilde{x}}{\underline{x}E\widetilde{y} + \alpha\left(E\widetilde{y} - E\widetilde{x}\right)}e - \frac{\alpha+\underline{x}}{\underline{x}}R\right] > 0,\\
q^* &= 0.
\end{aligned}$$

Proof. This follows from Lemma 4 and 12 in the Appendix. ■

For the regime (5), the landscape of optimal portfolios is enriched compared to (4). Optimal portfolios can again be illustrated graphically in the e/β-space. Note that the intercept for $\overline{e}^x$ and e^p is given by e^x_R. For e^q it is determined by e^z_R. Proposition 3a(i) shows that if income is sufficiently close to e_R^z or β is sufficiently high, then it is optimal to invest only in the z-asset. The intuition behind this result is the same as for Proposition 2(i). In both cases the marginal value of an increase of c^1 is very high relative to the marginal value of an increase of $E\widetilde{c}^2$. As a result, the only savings motive that is operational is to maintain habit consumption. This is achieved most efficiently by z-investments.

I discuss next the intuition behind the remaining parts of Proposition 3. Suppose that we start with an economy where no x-asset is available. Suppose further, that the gap between the marginal values of increases in c^1 and $E\widetilde{c}^2$ are sufficiently balanced. This requires that income is not too low and β not too high. As a result, it will be optimal to hold a portfolio with $q > 0$, $s > 0$. In particular, the ratio q/s will depend on e and β. Consider now what happens if the x-asset becomes available in this economy. If (5) holds, then it is possible to improve on the initial portfolio by reducing both, y- and z-investments, and increasing x-investments. This follows from the same logic as the opposite assertion for the regime of condition (4). (See the discussion following Proposition 2.) Eventually, the portfolio reallocation process is stopped as q or s hit on their lower bounds of zero. If q/s is high, then this will first occur for y-investments. As a result, we will end up with a portfolio $p > 0$, $q > 0$, $s = 0$. This corresponds to the portfolio in part (a)(ii) of Proposition 3. Next, it might happen that the lower bounds are attained simultaneously. In this case we have $p > 0$, $q = 0$, $s = 0$, corresponding to part (a)(iii). Finally, if q/s is low initially, then a portfolio with $p > 0$, $q = 0$, $s > 0$ will result that coincides with the portfolio indicated in part (b) of Proposition 3.

For the case

$$E\widetilde{y} - E\widetilde{z} = \frac{\underline{z}}{\underline{x}} (E\widetilde{y} - E\widetilde{x} \, , \tag{6}$$

the following observation can be inferred from Lemma 5, 6 and 12(b) in the Appendix. (It is instructive to study the proof of Lemma 5(iii).)

Remark 2 *Suppose that (6) holds. Then portfolios with $s > 0$ belong to the set of optimal choices if*

$$e > \overline{e}^x, \beta < \overline{\beta}^x \text{ or } e > \overline{e}^z, \beta < \overline{\beta}^z. \tag{$S^>$}$$

However, if ($S^>$) holds, then optimal portfolios are not unique. In particular, a portfolio with $s = 0$ belongs to the set of optimal choices if $\overline{\beta}^x \leq \beta < \overline{\beta}^z$ and $e > \overline{e}^z$. (Note that $\overline{\beta}^x < \overline{\beta}^z$.)

If ($S^>$) does not hold, then $s^* = 0$ and optimal choices coincide with the case where $E\widetilde{y} - E\widetilde{z} > \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$.

Overall, Propositions 1 to 3 and Remark 2 suggest that optimal risk management varies greatly between different income levels. Low-income earners are pure precautionary savers and do not value equity investments. Precautionary savings prevent that old-age consumption will ever fall below the habit level αc^1. High-income individuals have also a speculative motive of saving. (This requires that β is not too high.) With regard to pension policy, low-income earners will always be best off with a pure PAYGO system, as long as this has the lowest downside risk. For higher incomes, a PAYGO system is only beneficial if condition (4) holds. Moreover, the rate of precautionary savings, defined by $(p+q)/e$, decreases with income whenever $s > 0$. As a result, there might be disagreement between different income classes not only with respect to the desirability of a PAYGO system but also with respect to contribution rates. Whether we should expect that these disagreements will be substantial is an empirical question. Therefore, the next section evaluates (4) as well as optimal contribution rates from the perspective of historical and projected US data.

5 Quantitative Implications for Pension Design

Let us start with the evaluation of condition (4). Specifically, I will use this condition to infer a critical level $\underline{z}^{crit}$, such that it holds if and only $\underline{z}$ exceeds this particular level. Expected and minimum returns are derived in the same way as in Section 2. In particular, it is assumed that the expected value of $\tilde{z}$ is determined according to the long-term projections of the Social Security Administration, while the standard deviation corresponds to historical wage income growth. Note that, under this assumption, we are implicitly imposing a defined contribution PAYGO system where the risk of low wage income growth is entirely borne by the old generation. This contrasts with the current design of US Social Security, where there is some guarantee of a particular replacement rate.

With regard to the timing of the model, I assume that individuals work from age 21 until age 65. Furthermore, they will die at age 86.7. This corresponds to the average life-expectancy between men and women at age 65 in 2080.[12] I assume that individuals' investment horizon is given by the distance between the middle of the two life-cycle periods, measured in years, minus one. The latter subtraction refers to the assumption that savings are invested only at the end of a particular year during working life. But they are withdrawn at the beginning of a particular year during retirement. The resulting length of the investment horizon amounts to 32 years.

Figure 5 shows the results of the empirical evaluation of (4) where $\underline{x}$ and $\underline{z}$ correspond to alternative values of the inverse of the cumulative distribution functions of $\tilde{x}$ and $\tilde{z}$. The dotted line just represents $\underline{x}$. The solid line shows the resulting values for $\underline{z}^{crit}$. The dashed line depicts the corresponding values of $\underline{z}$. We infer from Figure 5 that (4) holds for the entire range of lower probabilities. Moreover, the distance between $\underline{z}$ and $\underline{z}^{crit}$ is quite large.[13] In light of Proposition 2, this makes the case for a PAYGO system. One might argue that focusing on the very lower end of the return distributions estimates downside risks too conservatively. Remember, however, from Section 2 that anyone not concerned about return percentiles below the fifth should never privately invest in bonds but only in stocks.

I turn next to the evaluation of optimal contribution rates. This requires a numerical specification of the preference parameters α, β, R. Concerning α, let us first specify a value that seems appropriate when comparing average *annual* consumption during retirement to average annual consumption during working life. Following Binswanger (2005), I set the fraction of annual active period consumption that individuals do not allow to be

[12]Source: Board of Trustees (2005).

[13]The same conclusion is obtained if $\underline{y}$ is specified according to cumulative probabilities rather than set to zero.

undercut during retirement equal to the average Social Security replacement rate. The latter amounts to .4 (see Munnell, 2004). This value has to be adjusted to take into account that the retirement period is typically shorter than working life. Without such an adjustment one would implicitly assume, that both periods last for an equal number of years. I arrive at a specification for α by multiplying the Social Security replacement rate by the ratio of the length of retirement to the length of working life. The resulting value amounts to .20.[14]

Concerning β, inspection of (2) shows that it just represents the inverse of a standard discount factor. A usual specification of the annual discount factor is .96. Raising this number to the power of 32 (the model's time horizon) and taking the inverse yields a raw estimate of β. I multiply this raw specification by the ratio of the length of retirement to the length of the active period, again, to account for the fact that the retirement period is typically shorter than working age. This yields an numerical specification for β of 7.4. The final parameter, R, is set to 25,000 real 2001 US dollars. It is shown in Binswanger (2005) that this value does a particularly good job in capturing the empirical income threshold beyond which individuals start to invest in stocks.

Figure 6 shows the cross-section of optimal PAYGO contribution and equity savings rates that are implied by the above parameter values. The upper diagram corresponds to the setting as it has been analyzed theoretically in Section 4. Here, $\underline{y}$ is set to zero. The lower diagram represents a situation where $\underline{y}$ corresponds to the third percentile of its distribution, which amounts to minus 42 percent.[15] For income levels below 35,000 real 2001 dollars, households are best off by having all their savings substituted by a PAYGO system. The optimal contribution rate amounts to 17.2 percent. According to the income quantile estimates of the US Census Bureau, this concerns the two lowest

[14]The specification of α by means of *average* life-expectancies is only valid under the availability of annuities. Otherwise, one would have to account for the risk of outliving one's assets such that a higher value of α would be appropriate.

[15]To determine optimal investments for the latter case it has first to be verified that bond investments are still zero. Under this prerequisite, investment choices are derived along the lines of Binswanger (2005).

income quantiles.[16] For higher income levels, the two diagrams in Figure 6 share the same qualitative properties. The quantitative implications differ, however. For $\underline{y} = 0$, PAYGO contribution rates remain quite high and equity savings rates are low at all income levels. In contrast, the two cross at 55,000 in the lower diagram. This roughly corresponds to the upper limit of the third income quintile. At this income level, both rates equal 10 percent. Thereafter, the optimal PAYGO rate gradually approaches zero and the equity savings rate moves gradually toward 26 percent.

At first sight, PAYGO contribution and equity savings rates seem rather high. After all, for the lower end of the income distribution the optimal PAYGO tax almost approaches the level it is projected to raise to for the US in the absence of benefit adjustments. Moreover, empirical savings rates for the two lowest income quintiles range only from 7 to 14 percent, even when accounting for Social Security saving (see Dynan et al., 2004). However, it should be noted, that this rates are low partly because low-income earners are subsidized within Social Security. They are also likely to be covered by other welfare programs. In contrast, the analysis here abstracts from any redistribution. For higher income levels the total savings rates obtained fit well into the range of empirical estimates.

Should the US thus keep its current replacement rates, raising contributions to 17 or 18 percent? The model implies that we need to differentiate. Low-income earners would be better off with such a non-reformed system than with the individual account system proposed by President Bush. In contrast, earners of higher incomes are expected to gain by the reform. Households located at the very right end of the income distribution would even prefer to abandon any PAYGO component. The bottom line of the analysis is quite clear-cut. Even apart from redistribution, optimal contribution rates of a particular pension component depend quite strongly on income. Thus, whenever it is feasible to cover low-income earners by a pure PAYGO system, the middle class by a mixed system and rich households by a purely funded system (or no system at all), the model suggests that this would be beneficial.

The analysis of this paper has abstracted from the existence of a risk-free asset. Since

[16]See http:/www.census.gov/hhes/income/histinc/h01ar.html, Historical Income Table H-1.

several years, such asset are available for the US in the form of TIPS (Treasury Inflation-Protected Securities). Is a PAYGO system thus obsolete? The analysis of Section 4 can be applied to gain some insights into the answer of this question. Indeed, a PAYGO system should be replaced by TIPS for low-income earners whenever the returns of the latter exceed minimum returns for the former. For higher income levels, a condition similar to (4) will decide whether or not TIPS are more beneficial than a PAYGO system. Of course, this consideration is limited to the comparison of steady sates as the entire analysis of this paper.

There are three problems that may render TIPS less attractive for pension systems to rely on than it may have seemed at first sight. First, there is comparably little historical knowledge with respect to the level of returns. Current long-term returns range from 1.8 to 2.2 percent for the US.[17] A second problem is that even TIPS are not completely risk-free. The reason is that their nominal returns are adjusted according to some broad average increase in prices. This might not correspond to closely to the costs of living of any particular household. A third and probably most severe problem is that the reliance on such bonds requires a sufficient amount of debt financing for the entire future. This might have adverse effects on spending behavior of government agents.

6 Conclusion

The paper applies a new lexicographic life-cycle model to the study of optimal pension design. It explores how a pension system should be composed of a PAYGO and alternative funded components. Low-income earners are always best off with a pure PAYGO system. For high-income earners a simple and intuitive condition is derived that decides whether a PAYGO system will be beneficial at all. The calibrations in Section 5 suggest that this is, indeed, the case for empirically relevant specifications of the model. High-income earners will always complement PAYGO "savings" by financial market investments. Their optimal PAYGO contribution rate is considerably less than for low-income earners.

[17] Source: Federal Reserve Statistics and Department of the Treasury.

The paper makes a case for the fact that optimal risk-management of old-age provision is quite sensitive to the income level of covered households. Income heterogeneity is an aspect of pension design that has attracted relatively little attention, apart from redistributive issues. This may be partly explained by the fact that the theoretical models that underly almost all applied studies of risk-management do not leave any room for "structural breaks" in risk aversion along the income distribution. However, the empirical cross-section of portfolio choices strongly suggests that such breaks should be taken into account.

Future research will have to take into account the benefits from diversification between different types of investments by capturing risk aversion in a more general form. Second, it is important to examine transitional issues. With respect to income heterogeneity, it is crucial to explore how the recommendations of the paper are implementable from a pragmatic point of view. Finally, the framework might be extended to a multiperiod setting to study the optimal profile of contribution rates over the working age.

Appendix

The Kuhn-Tucker conditions for the program (3) are given by

$$\frac{E\widetilde{x}}{pE\widetilde{x}+qE\widetilde{z}+sE\widetilde{y}}-\frac{\beta}{e-p-q-s-R}+(\alpha+\underline{x})\,\lambda+\nu=0, \tag{7}$$

$$\frac{E\widetilde{z}}{pE\widetilde{x}+qE\widetilde{z}+sE\widetilde{y}}-\frac{\beta}{e-p-q-s-R}+(\alpha+\underline{z})\,\lambda+\xi=0, \tag{8}$$

$$\frac{E\widetilde{y}}{pE\widetilde{x}+qE\widetilde{z}+sE\widetilde{y}}-\frac{\beta}{e-p-q-s-R}+\alpha\lambda+\mu=0, \tag{9}$$

$$\lambda\geq 0,\, p\underline{x}+q\underline{z}-\alpha\,(e-p-q-s)\geq 0,\, \lambda\,[p\underline{x}+q\underline{z}-\alpha\,(e-p-q-s)]=0, \tag{10}$$

$$\mu\geq 0,\, s\geq 0,\, \mu s=0, \tag{11}$$

$$\nu\geq 0,\, p\geq 0,\, \nu p=0, \tag{12}$$

$$\xi\geq 0,\, q\geq 0,\, \xi q=0. \tag{13}$$

Remember from Section 4.3, in particular, from Proposition 4, that the program (3) is well defined if and only if $e > e_R^{\underline{z}} \equiv \frac{\alpha+\underline{z}}{\underline{z}}e$ since $c^1 > R$ is feasible, conditional on $c^{2\min} \geq \alpha c^1$, if and only if $e > e_R^{\underline{z}}$. Note that an optimal choice exists for which the conditions (7)-(13) are necessary and sufficient since the program (3) is concave.[18] Lemma 1 strengthens this observation.

Lemma 1 *If* $E\widetilde{y}-E\widetilde{z}\neq\frac{\underline{z}}{\underline{x}}\left(E\widetilde{y}-E\widetilde{x}\right)$ *then optimal choices are unique.*

Proof. From (7)-(9) it follows that

$$\lambda=\frac{E\widetilde{y}-E\widetilde{x}}{\underline{x}E\widetilde{c}^2}+\frac{\mu-\nu}{\underline{x}}=\frac{E\widetilde{y}-E\widetilde{z}}{\underline{z}E\widetilde{c}^2}+\frac{\mu-\xi}{\underline{z}}, \tag{14}$$

where $E\widetilde{c}^2\equiv pE\widetilde{x}+qE\widetilde{z}+sE\widetilde{y}$. For $c^1>0$ the constraint $p\underline{x}+q\underline{z}-\alpha\,(e-p-q-s)\geq 0$, which is equivalent to $c^{2\min}\geq\alpha c^1$, can only be fulfilled if $p>0$ or $q>0$ from Assumption 3. Therefore $\nu=0$ or $\xi=0$ by (12), (13). Since $\mu\geq 0$ it follows that $\lambda>0$. Therefore, the constraint $p\underline{x}+q\underline{z}-\alpha\,(e-p-q-s)\geq 0$ is binding from (10). Thus, the optimal solution of (3) is identical to the solution of a similar program where this constraint is

[18] The existence of an optimum follows from the fact that the second argument of (2) is continuous and the feasible set is closed and bounded.

directly required to hold with equality. Solving $p\underline{x} + q\underline{z} = \alpha (e - p - q - s)$ for q we obtain $q = \frac{\alpha}{\alpha+\underline{z}}e - \frac{\alpha+\underline{x}}{\alpha+\underline{z}}p - \frac{\alpha}{\alpha+\underline{z}}s$. Using this and the budget constraints c^1 and $E\widetilde{c}^2$ can be expressed in terms of p, s such that the optimal decision is determined by

$$\begin{aligned} &\max_{p,s} \ln\left[\frac{\alpha}{\alpha+\underline{z}}E\widetilde{z}e + \left(E\widetilde{x} - \frac{\alpha+\underline{x}}{\alpha+\underline{z}}E\widetilde{z}\right)p + \left(E\widetilde{y} - \frac{\alpha}{\alpha+\underline{z}}E\widetilde{z}\right)s\right] + \\ &\qquad \beta \ln\left[\frac{\underline{z}}{\alpha+\underline{z}}e - \frac{\underline{z}-\underline{x}}{\alpha+\underline{z}}p - \frac{\underline{z}}{\alpha+\underline{z}}s - R\right] \quad \text{s.t.} \\ &p \geq 0,\ s \geq 0,\ \frac{\alpha}{\alpha+\underline{z}}e - \frac{\alpha+\underline{x}}{\alpha+\underline{z}}p - \frac{\alpha}{\alpha+\underline{z}}s \geq 0. \end{aligned}$$

Calculating the second derivatives of the objective function it turns out that the determinant of the Hessian matrix is strictly positive if $E\widetilde{y} - E\widetilde{z} \neq \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$ and zero otherwise. Thus, for $E\widetilde{y} - E\widetilde{z} \neq \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$ the objective function is strictly concave in p, s. From this and the linearity of the constraints it follows that any optimal solution is unique. ■

The following definitions are useful for the characterization of the solution.

$$\begin{aligned} \overline{e}^x &\equiv \frac{\alpha+\underline{x}}{\underline{x}} \frac{\underline{x}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{x})}{\underline{x}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{x}) - \alpha\beta E\widetilde{x}} R \\ \overline{e}^z &\equiv \frac{\alpha+\underline{z}}{\underline{z}} \frac{\underline{z}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{z})}{\underline{z}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{z}) - \alpha\beta E\widetilde{z}} R \\ e^p &\equiv \frac{[(\alpha+\underline{z})E\widetilde{x} - (\alpha+\underline{x})E\widetilde{z}](\alpha+\underline{x})}{[\underline{x}(\alpha+\underline{z}) - \alpha\beta(\underline{z}-\underline{x})]E\widetilde{x} - \underline{x}(\alpha+\underline{x})E\widetilde{z}} R, \\ e^q &\equiv \frac{[(\alpha+\underline{z})E\widetilde{x} - (\alpha+\underline{x})E\widetilde{z}](\alpha+\underline{z})}{\underline{z}(\alpha+\underline{z})E\widetilde{x} - [\underline{z}(\alpha+\underline{x}) + \alpha\beta(\underline{z}-\underline{x})]E\widetilde{z}} R, \\ \overline{\beta}^x &\equiv \frac{\alpha+\underline{x}}{\alpha}\frac{E\widetilde{y}}{E\widetilde{x}} - 1, \\ \overline{\beta}^z &\equiv \frac{\alpha+\underline{z}}{\alpha}\frac{E\widetilde{y}}{E\widetilde{z}} - 1, \\ \beta^p &\equiv \frac{\underline{x}}{\alpha(\underline{z}-\underline{x})}\left[\alpha+\underline{z} - (\alpha+\underline{x})\frac{E\widetilde{z}}{E\widetilde{x}}\right], \\ \beta^q &\equiv \frac{\underline{z}}{\alpha(\underline{z}-\underline{x})}\left[(\alpha+\underline{z})\frac{E\widetilde{x}}{E\widetilde{z}} - \alpha - \underline{x}\right]. \end{aligned}$$

Remark 4 *By direct calculation the following inequalities can be shown to hold:*

(i) $e_R^z < \overline{e}^x$, $e_R^z < \overline{e}^z$,

(ii) $e^q < e^p$ *for* $\beta \leq \beta^p$,

(iii) $\beta^p < \beta^q$,

(iv) $\overline{\beta}^x < \overline{\beta}^z$,

(v) $\beta^p \gtreqless \overline{\beta}^x$ *if and only if* $E\widetilde{y} - E\widetilde{z} \gtreqless \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$,

(vi) $\beta^q \lesseqgtr \overline{\beta}^z$ *if and only if* $E\widetilde{y} - E\widetilde{z} \lesseqgtr \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$.

The following lemmas characterize the optimal solution of the program (3). It is convenient to first analyze the cases where $s^* > 0$ and then to continue with cases where $s^* = 0$.

Lemma 2 *Assume* $s^* > 0$*. Then*

(i) $q^* > 0$ *and* $p^* = 0$ *if* $E\widetilde{y} - E\widetilde{z} < \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$.

(ii) $p^* > 0$ *and* $q^* = 0$ *if* $E\widetilde{y} - E\widetilde{z} > \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$.

(iii) $p^* > 0$ *and* $q^* > 0$ *only if* $E\widetilde{y} - E\widetilde{z} = \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$.

Proof. Proof of *(i)*. By hypothesis $s^* > 0$ and $\frac{E\widetilde{y} - E\widetilde{x}}{\underline{x}} > \frac{E\widetilde{y} - E\widetilde{z}}{\underline{z}}$. Thus, it follows from (14) and (11) that $\frac{\xi}{\underline{z}} < \frac{\nu}{\underline{x}}$. Since, for $c^1 > 0$, (10) does only hold if $p > 0$ or $q > 0$ it follows from (12), (13) that $\nu = 0$ or $\xi = 0$. From (13) we must therefore have $\nu > 0$ and hence $p^* = 0$ and $q^* > 0$.

The proof of *(ii)* is similar.

Proof of *(iii)*. $p^* > 0$, $q^* > 0$ require $\nu = \xi = 0$ from (12), (13). It follows from (14) that this implies $E\widetilde{y} - E\widetilde{z} = \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$. ■

Lemma 3 *Assume* $e > e^z_R$ *and* $E\widetilde{y} - E\widetilde{z} < \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$*. If*

$$e > \overline{e}^z \text{ and } \beta < \overline{\beta}^z. \tag{SZ}$$

then

$$c^{1*} = \frac{1}{1+\beta}R + \frac{\beta}{1+\beta}\frac{\underline{z}E\widetilde{y}}{\underline{z}E\widetilde{y} + \alpha\left(E\widetilde{y} - E\widetilde{z}\right)}e, \tag{15}$$

$$q^* = \frac{\alpha}{\underline{z}}\left[\frac{1}{1+\beta}R + \frac{\beta}{1+\beta}\frac{\underline{z}E\widetilde{y}}{\underline{z}E\widetilde{y} + \alpha\left(E\widetilde{y} - E\widetilde{z}\right)}e\right], \tag{16}$$

$$s^* = \frac{1}{1+\beta}\left[\frac{\left(\alpha + \underline{z}\right)E\widetilde{y} - \alpha\left(1+\beta\right)E\widetilde{z}}{\underline{z}E\widetilde{y} + \alpha\left(E\widetilde{y} - E\widetilde{z}\right)}e - \frac{\alpha + \underline{z}}{\underline{z}}R\right] > 0, \tag{17}$$

$$p^* = 0. \tag{18}$$

Proof. (SZ) implies that the expression for s^* is strictly positive. Thus, $\mu = 0$ by (11). Since $q^* > 0$ we have $\xi = 0$ from (13). Check that (8), (9) hold if $\lambda = \frac{E\widetilde{y} - E\widetilde{z}}{\underline{z}E\widetilde{c}^2} > 0$, where $E\widetilde{c}^2 \equiv pE\widetilde{x} + qE\widetilde{z} + sE\widetilde{y}$. Moreover, $p\underline{x} + q\underline{z} - \alpha(e - p - q - s) = 0$ such that (10) holds. From (7) $\nu = \frac{\beta}{c^1 - R} - \frac{E\widetilde{x}}{E\widetilde{c}^2} - (\alpha + \underline{x})\lambda$. Inserting for λ, c^1, p, q, s and rearranging shows that $\nu \geq 0$ if and only if

$$\left[E\widetilde{y} - E\widetilde{x} - \frac{\underline{x}}{\underline{z}}(E\widetilde{y} - E\widetilde{z})\right]\left[\frac{\underline{z}E\widetilde{y}}{\underline{z}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{z})}e - R\right] \geq 0.$$

By hypothesis $E\widetilde{y} - E\widetilde{x} - \frac{\underline{x}}{\underline{z}}(E\widetilde{y} - E\widetilde{z}) > 0$. Therefore $\nu \geq 0$ if and only if $\frac{\underline{z}E\widetilde{y}}{\underline{z}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{z})}e \geq R$ or $e \geq \frac{\underline{z}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{z})}{\underline{z}E\widetilde{y}}R$. It can be checked that this inequality is implied by (SZ). Thus, (12) holds. We conclude that all Kuhn-Tucker conditions are fulfilled. ■

Lemma 4 *Assume* $e > e_R^z$ *and* $E\widetilde{y} - E\widetilde{z} > \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$. *If*

$$e > \overline{e}^x \text{ and } \beta < \overline{\beta}^x. \tag{SX}$$

then

$$c^{1*} = \frac{1}{1+\beta}R + \frac{\beta}{1+\beta}\frac{\underline{x}E\widetilde{y}}{\underline{x}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{x})}e, \tag{19}$$

$$p^* = \frac{\alpha}{\underline{x}}\left[\frac{1}{1+\beta}R + \frac{\beta}{1+\beta}\frac{\underline{x}E\widetilde{y}}{\underline{x}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{x})}e\right], \tag{20}$$

$$s^* = \frac{1}{1+\beta}\left[\frac{(\alpha + \underline{x})E\widetilde{y} - \alpha(1+\beta)E\widetilde{x}}{\underline{x}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{x})}e - \frac{\alpha + \underline{x}}{\underline{x}}R\right] > 0, \tag{21}$$

$$q^* = 0. \tag{22}$$

Proof. The proof is similar to the previous lemma. Check that (7), (9) hold for $\lambda = \frac{E\widetilde{y} - E\widetilde{x}}{\underline{x}E\widetilde{c}^2}$. From (8) $\xi = \frac{\beta}{c^1 - R} - \frac{E\widetilde{z}}{E\widetilde{c}^2} - (\alpha + \underline{z})\lambda$. Inserting for λ, c^1, p, q, s and rearranging shows that $\xi \geq 0$ if and only if

$$\left[E\widetilde{y} - E\widetilde{z} - \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})\right]\left[\frac{\underline{x}E\widetilde{y}}{\underline{x}E\widetilde{y} + \alpha(E\widetilde{y} - E\widetilde{x})}e - R\right] \geq 0.$$

By hypothesis the term in the first bracket is positive. Check that (SX) implies that the expression in the second bracket is also positive. ■

Lemma 5 *Assume* $E\widetilde{y} - E\widetilde{z} = \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$.

(i) *If (SZ) holds then the choice (15)-(18) is optimal.*

(ii) *If (SX) holds then the choice (19)-(22) is optimal.*

(iii) *There exists an optimal choice with* $p > 0$, $q > 0$, $s > 0$ *if and only if the choice (15)-(18) is optimal and (SZ) holds.*

(iv) *If (SZ) holds but (SX) does not hold then there exists an optimal portfolio with* $s = 0$. *(Note that (SZ) holds but (SX) does not hold whenever* $\overline{\beta}^x \leq \beta < \overline{\beta}^z$ *and* $e > \overline{e}^z$*.)*

Proof. Proof of (i). Proceeding in the same way as in the proof of Lemma 3 it is shown that if (SZ) holds then the choice (15)-(18) fulfils all Kuhn-Tucker conditions also for the case where $E\widetilde{y} - E\widetilde{z} = \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$. (Note that this implies $\nu = 0$).

The proof of (ii) is similar to that of Lemma 4.

Proof of (iii). Consider a portfolio with $p > 0$, $q > 0$, $s > 0$. Consider a reallocation with $\Delta p < 0$, $\Delta q = -\frac{x}{\underline{z}}\Delta p > 0$, $\Delta s = -\frac{1}{E\widetilde{y}}\left(E\widetilde{x} - \frac{x}{\underline{z}}E\widetilde{z}\right)\Delta p > 0$. $E\widetilde{y} - E\widetilde{z} = \frac{z}{\underline{x}}\left(E\widetilde{y} - E\widetilde{x}\right)$ implies that $\Delta p + \Delta q + \Delta s = 0$. Furthermore, we have $\Delta\widetilde{c}^{2\,\min} = \Delta E\widetilde{c}^2 = 0$. Thus, according to the preferences (2) there must be indifference between the original and the reallocated portfolio. If $\Delta p = -p$ the reallocation leads to a portfolio with $p = 0$, $q > 0$, $s > 0$. Obviously, if the original portfolio was optimal then the new one must be so. Thus, the new portfolio coincides with the one given by (16)-(18). To see this note that any optimal choice with $p = 0$, $q > 0$, $s > 0$ is determined as the unique solution of the system of the three equations (8), (9) and $q\underline{z} = \alpha\left(e - q - s\right)$ in the three unknowns λ, p, q. (Use (11), (13). Check (10).) Furthermore, the expression for s^* is *strictly* positive only if (SZ) holds.

The converse is proved by noting that if the portfolio (16)-(18) is optimal then there exists an inverse reallocation with $\Delta p > 0$, $\Delta q < 0$, $\Delta s < 0$, $\Delta p + \Delta q + \Delta s = 0$ which leads to an optimal portfolio with $p > 0$, $q > 0$, $s > 0$. The expression for s^* is *strictly* positive if (SZ) holds such that a reallocation with $\Delta p > 0$, $\Delta q < 0$, $\Delta s < 0$ is feasible.

Proof of (iv). Since (SZ) holds the portfolio (16)-(18) is optimal. From the proof of (iii) there exists a portfolio reallocation with $\Delta p > 0$, $\Delta q < 0$, $\Delta s < 0$ leading to other optimal portfolios. In particular, by arguments similar to those in the proof of (*iii*), the reallocation leads to the portfolio (20)-(22) for Δq equal to the negative of the expression for q^* in (16). This portfolio is feasible, and hence belongs to the set of optimal portfolios, if and only if the expression for s^* in (21) is nonnegative. Since (SX) does not hold the expression for s^* in (21) is nonpositive. It follows from this that there exists there exists a reallocation with Δq greater or equal to the negative of the expression for q^* in (16), leading from (16)-(18) to an optimal portfolio with $s = 0$. ■

Lemma 6

(*i*) *If* $E\widetilde{y} - E\widetilde{z} < \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$ *then* $s^* > 0$ *if and only if (SZ) holds.*

(*ii*) *If* $E\widetilde{y} - E\widetilde{z} > \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$ *then* $s^* > 0$ *if and only if (SX) holds.*

(*iii*) *If* $E\widetilde{y} - E\widetilde{z} = \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$ *then a portfolio with* $s > 0$ *is optimal if and only if (SX) or (SZ) hold.*

Proof. Proof of (*i*). From Lemma 2(*i*) $s^* > 0$ only if $p^* = 0$, $q^* > 0$. From (11), (13) $s^* > 0$, $q^* > 0$ only if $\mu = \xi = 0$. Thus, any optimal choice with $s^* > 0$ is determined as the solution of the system of the three equations (8), (9) and $p\underline{x} + q\underline{z} = \alpha(e - p - q - s)$ in the three unknowns λ, p, q (note that $\lambda > 0$ for an optimal choice from (14)). Since the solution of this system of equations is unique it follows that $s^* > 0$, $q^* > 0$ is optimal only if the optimal choice is determined by (15)-(18). The expression for s^* in (17) is strictly positive only if (SZ) holds. Conversely, if (SZ) holds then the choice (15)-(18) is optimal from Lemma 3 and the expression for s^* is strictly positive.

(*ii*) is proved in the same way, using Lemma 2(*ii*).

Proof of (*iii*). If (SZ) or (SX) hold then the choices (15)-(18) or (19)-(22) are optimal from Lemma 5(*i*) and (*ii*), respectively. The expressions for s^* in (17), (21) are strictly positive if (SZ) or (SX) hold, respectively. Conversely, if both, (SZ) and (SX) do not hold, then the expression for s^* in (17), (21) are nonpositive. Moreover, from Lemma 5(*iii*) choices with $p > 0$, $q > 0$, $s > 0$ are optimal only if (SZ) holds. This yields the result. ■

Lemma 7 *Assume* $e > e_R^z$ *and* $s^* = 0$.

(i) If $p^* = 0$ *then* $c^{1*} = \frac{\underline{z}}{\alpha+\underline{z}} e$, $q^* = \frac{\alpha}{\alpha+\underline{z}} e$.

(ii) If $q^* = 0$ *then* $c^{1*} = \frac{\underline{x}}{\alpha+\underline{x}} e$, $p^* = \frac{\alpha}{\alpha+\underline{x}} e$.

(iii) If $p^* q^* > 0$ *then*

$$\begin{aligned} c^{1*} &= \frac{1}{1+\beta} R + \frac{\beta}{1+\beta} \frac{\underline{z} E\widetilde{x} - \underline{x} E\widetilde{z}}{(\alpha+\underline{z}) E\widetilde{x} - (\alpha+\underline{x}) E\widetilde{z}} e, \\ p^* &= \frac{1}{(1+\beta)(\underline{z}-\underline{x})} \left[\frac{\underline{z}(\alpha+\underline{z}) E\widetilde{x} - [\underline{z}(\alpha+\underline{x}) + \alpha\beta(\underline{z}-\underline{x})] E\widetilde{z}}{(\alpha+\underline{z}) E\widetilde{x} - (\alpha+\underline{x}) E\widetilde{z}} e - (\alpha+\underline{z}) R \right] \\ q^* &= \frac{1}{(1+\beta)(\underline{z}-\underline{x})} \left[\frac{[\alpha\beta(\underline{z}-\underline{x}) - \underline{x}(\alpha+\underline{z})] E\widetilde{x} + \underline{x}(\alpha+\underline{x}) E\widetilde{z}}{(\alpha+\underline{z}) E\widetilde{x} - (\alpha+\underline{x}) E\widetilde{z}} e + (\alpha+\underline{x}) R \right]. \end{aligned}$$

Proof. Proof of (i). From (14) $\lambda > 0$ for an optimal choice such that (10) implies $q\underline{z} = \alpha(e-q)$. Solving this for q and using the budget constraint yields the result.

(ii) is proved similarly.

Proof of (iii). $p^* > 0$, $q^* > 0$ implies $\nu = \xi = 0$. With this, (7), (8) imply $\lambda = \frac{E\widetilde{x} - E\widetilde{z}}{\underline{z}-\underline{x}} \frac{1}{E\widetilde{c}^2} > 0$ where $E\widetilde{c}^2 \equiv pE\widetilde{x} + qE\widetilde{z}$. It follows that

$$p\underline{x} + q\underline{z} = \alpha(e-p-q). \tag{23}$$

Inserting the expression for λ, (7), (8), (23) represent a system of three equations in the three unknowns c^1, p, q. Solving the system yields the expressions for c^{1*}, p^*, q^*. ■

Lemma 8 *Assume* $e > e_R^z$ *and* $s^* = 0$.

(i) If $c^1 = \frac{\underline{z}}{\alpha+\underline{z}} e$, $q = \frac{\alpha}{\alpha+\underline{z}} e$, $p = 0$ *and (SZ) does not hold then all Kuhn-Tucker conditions are fulfilled if and only if*

$$\beta \geq \beta^q \text{ or } e \leq e^q. \tag{NSZ}$$

(ii) If $c^1 = \frac{\underline{x}}{\alpha+\underline{x}} e$, $p = \frac{\alpha}{\alpha+\underline{x}} e$, $q = 0$ *then all Kuhn-Tucker conditions are fulfilled only if*

$$\beta < \beta^p \text{ and } e \geq e^p. \tag{NSX}$$

(iii) If $c^1 = \frac{\underline{x}}{\alpha+\underline{x}} e$, $p = \frac{\alpha}{\alpha+\underline{x}} e$, $q = 0$ *and (SX) does not hold then all Kuhn-Tucker conditions are fulfilled if and only if (NSX) holds.*

Proof. Proof of (i). Since $q>0$, (13) requires $\xi=0$. Using this, solve (8) for λ to obtain $\lambda=\frac{\frac{(\alpha\beta-\underline{z})}{\alpha+\underline{z}}e+R}{\alpha(c^1-R)e}$. $e>e_R^z$ implies $c^1>R$. Thus, the denominator is strictly positive. The numerator is positive if $\beta\geq\frac{\underline{z}}{\alpha}$ or, for $\beta<\frac{\underline{z}}{\alpha}$, if $e<\frac{\alpha+\underline{z}}{\underline{z}-\alpha\beta}R$. Check that $\overline{\beta}^z>\frac{\underline{z}}{\alpha}$ and, for $\beta<\frac{\underline{z}}{\alpha}$, we have $\overline{e}^z<\frac{\alpha+\underline{z}}{\underline{z}-\alpha\beta}R$. Thus, $\lambda>0$ if (SZ) does not hold. Since $q\underline{z}=\alpha c^1$ it follows that (10) is fulfilled. From (7) $\nu=\frac{\beta}{c^1-R}-\frac{E\widetilde{x}}{qE\widetilde{z}}-(\alpha+\underline{x})\lambda$. Inserting for λ,c^1,q reveals that $\nu\geq0$ if and only if

$$\frac{\left[\alpha\beta\left(\underline{z}-\underline{x}\right)+\underline{z}\left(\alpha+\underline{x}\right)\right]E\widetilde{z}-\underline{z}\left(\alpha+\underline{z}\right)E\widetilde{x}}{\left(\alpha+\underline{z}\right)^2}e+\left[E\widetilde{x}-\frac{\alpha+\underline{x}}{\alpha+\underline{z}}E\widetilde{z}\right]R\geq0.$$

Check that this inequality and thus (12) is fulfilled if and only if (NSZ) holds. Finally, (9) and (11) must hold for some $\mu\geq0$ since, by assumption, $s^*=0$.

(ii) and (iii) are proved in a similar way. Note that $\xi\geq0$ if and only if

$$\frac{\left[\underline{x}\left(\alpha+\underline{z}\right)-\alpha\beta\left(\underline{z}-\underline{x}\right)\right]E\widetilde{x}-\underline{x}\left(\alpha+\underline{x}\right)E\widetilde{z}}{\left(\alpha+\underline{x}\right)^2}e-\left[\frac{\alpha+\underline{z}}{\alpha+\underline{x}}E\widetilde{x}-E\widetilde{z}\right]R\geq0.$$

■

Lemma 9 *Assume $e>e_R^z$. Then*

(i) *$p^*>0$, $q^*>0$ only if $E\widetilde{y}-E\widetilde{z}\geq\frac{\underline{z}}{\underline{x}}\left(E\widetilde{y}-E\widetilde{x}\right)$*

(ii) *If $s^*=0$ then all Kuhn-Tucker conditions hold for*

$$\begin{aligned}
c^1 &= \frac{1}{1+\beta}R+\frac{\beta}{1+\beta}\frac{\underline{z}E\widetilde{x}-\underline{x}E\widetilde{z}}{\left(\alpha+\underline{z}\right)E\widetilde{x}-\left(\alpha+\underline{x}\right)E\widetilde{z}}e,\\
p &= \frac{1}{\left(1+\beta\right)\left(\underline{z}-\underline{x}\right)}\left[\frac{\underline{z}\left(\alpha+\underline{z}\right)E\widetilde{x}-\left[\underline{z}\left(\alpha+\underline{x}\right)+\alpha\beta\left(\underline{z}-\underline{x}\right)\right]E\widetilde{z}}{\left(\alpha+\underline{z}\right)E\widetilde{x}-\left(\alpha+\underline{x}\right)E\widetilde{z}}e-\left(\alpha+\underline{z}\right)R\right],\\
q &= \frac{1}{\left(1+\beta\right)\left(\underline{z}-\underline{x}\right)}\left[\frac{\left[\alpha\beta\left(\underline{z}-\underline{x}\right)-\underline{x}\left(\alpha+\underline{z}\right)\right]E\widetilde{x}+\underline{x}\left(\alpha+\underline{x}\right)E\widetilde{z}}{\left(\alpha+\underline{z}\right)E\widetilde{x}-\left(\alpha+\underline{x}\right)E\widetilde{z}}e+\left(\alpha+\underline{x}\right)R\right].
\end{aligned}$$

if and only if (NSX) and (NSZ) do not hold.

Proof. Proof of (i). $p>0$, $q>0$ requires $\nu=\xi=0$ from (12), (13). From (14) it follows that $\mu=\frac{\underline{x}(E\widetilde{y}-E\widetilde{z})-\underline{z}(E\widetilde{y}-E\widetilde{x})}{\underline{z}-\underline{x}}\frac{1}{E\widetilde{c}^2}$ in this case. Thus $\mu\geq0$ only if $E\widetilde{y}-E\widetilde{z}\geq\frac{\underline{z}}{\underline{x}}\left(E\widetilde{y}-E\widetilde{x}\right)$.

Proof of (ii). The expression for p is strictly positive if and only if $e>e^q$ and $\beta<\beta^q$. The expression for q is strictly positive if and only if $e<e^p$ or $\beta\geq\beta^p$. We conclude that

(12), (13) hold for $\nu = \xi = 0$ if and only if (NSX) and (NSZ) do not hold. Check that (7) (8) hold for $\lambda = \frac{E\widetilde{x} - E\widetilde{z}}{(\underline{z} - \underline{x}) E\widetilde{c}^2} > 0$ and $\nu = \xi = 0$. Since $p\underline{x} + q\underline{z} - \alpha(e - p - q - s) = 0$ (10) is fulfilled. Moreover, (9) and (11) hold for some $\mu \geq 0$ since, by assumption, $s^* = 0$. ■

Lemma 10 *Assume $E\widetilde{y} - E\widetilde{z} \geq \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$. If $s^* = 0$ and (NSZ) holds then (SZ) does not hold.*

Proof. If (NSZ) holds then only the choice $c^1 = \frac{\underline{z}}{\alpha + \underline{z}} e$, $q = \frac{\alpha}{\alpha + \underline{z}} e$, $p = 0$ can be optimal from Lemma 8(*ii*), 9(*ii*) and 7. (Note that the intersection of (NSX) and (NSZ) is empty because of Remark 4(*ii*) and (*iii*).) From (13) $\xi = 0$. Using this solve (8) for λ to obtain $\lambda = \frac{1}{\alpha + \underline{z}} \left[\frac{\beta}{c^1 - R} - \frac{1}{q} \right]$. From (9) we obtain $\mu = \frac{\beta}{c^1 - R} - \frac{E\widetilde{y}}{qE\widetilde{z}} - \alpha\lambda$. Inserting for λ, c^1, q it can be shown that $\mu \geq 0$ only if (SZ) does not hold. By assumption $s^* = 0$. This implies that (9), (11) hold for some $\mu \geq 0$. We conclude from this (SZ) must not hold. ■

Lemma 11 *Assume $e > e_R^z$ and $E\widetilde{y} - E\widetilde{z} < \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$. If (SZ) does not hold then $c^{1*} = \frac{\underline{z}}{\alpha + \underline{z}} e$, $q^* = \frac{\alpha}{\alpha + \underline{z}} e$, $p^* = 0$, $s^* = 0$.*

Proof. I show first that if $E\widetilde{y} - E\widetilde{z} < \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$ then $\overline{e}^z \leq e^q$ for $\beta < \beta^q$. Suppose not. Then there exist some values $e' < \overline{e}^z$ and β' such that either $\beta' < \beta^p$ and $e^q < e' < e^p$ or $\beta^p \leq \beta' < \beta^q$ and $e' > e^q$. To see this note that $\overline{e}^z > 0$ for $\beta \leq \overline{\beta}^z$. Furthermore, $\beta^p < \beta^q$, and, for $\beta < \beta^p$, we have $0 < e^q < e^p$ from the definition of e^q and by Remark 4(*ii*) and (*iii*). For $e' < \overline{e}^z$ we have $s^* = 0$ from Lemma 6(*i*). From Lemma 9(*i*) either $p^* = 0$ or $q^* = 0$. In both cases, whether $\beta' < \beta^p$ and $e^q < e' < e^p$ or whether $\beta^p \leq \beta' < \beta^q$ and $e' > e^q$, neither $p^* = 0$ nor $q^* = 0$ can be optimal because of Lemma 7, 8(*i*) and (*ii*). (Use $\beta^p < \beta^q$.) Since an optimal solution of (3) exists for $e > e_R^z$ for which the Kuhn-Tucker conditions (7)–(13) are necessary and sufficient this is a contradiction. We conclude that $\overline{e}^z \leq e^q$ for $\beta < \beta^q$. From Remark 4(*vi*) $\beta^q < \overline{\beta}^z$. Hence, if (SZ) does not hold then (NSZ) holds. The result follows then from Lemma 6(*i*) and 8(*i*). ■

Lemma 12 *Assume $e > e_R^z$.*

(*a*) *If $E\widetilde{y} - E\widetilde{z} > \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$ and (SX) does not hold then $s^* = 0$. Moreover,* *(i) If (NSZ) holds then* $c^{1*} = \frac{\underline{z}}{\alpha+\underline{z}}e$, $q^* = \frac{\alpha}{\alpha+\underline{z}}e$, $p^* = 0$.

(ii) If (NSX) holds then $c^{1*} = \frac{\underline{x}}{\alpha+\underline{x}}e$, $p^* = \frac{\alpha}{\alpha+\underline{x}}e$, $q^* = 0$.

(iii) If (NSX) and (NSZ) do not hold then

$$\begin{aligned}
c^1 &= \frac{1}{1+\beta}R + \frac{\beta}{1+\beta}\frac{\underline{z}E\widetilde{x} - \underline{x}E\widetilde{z}}{(\alpha+\underline{z})E\widetilde{x} - (\alpha+\underline{x})E\widetilde{z}}e, \\
p &= \frac{1}{(1+\beta)(\underline{z}-\underline{x})}\left[\frac{\underline{z}(\alpha+\underline{z})E\widetilde{x} - [\underline{z}(\alpha+\underline{x}) + \alpha\beta(\underline{z}-\underline{x})]E\widetilde{z}}{(\alpha+\underline{z})E\widetilde{x} - (\alpha+\underline{x})E\widetilde{z}}e - (\alpha+\underline{z})R\right], \\
q &= \frac{1}{(1+\beta)(\underline{z}-\underline{x})}\left[\frac{[\alpha\beta(\underline{z}-\underline{x}) - \underline{x}(\alpha+\underline{z})]E\widetilde{x} + \underline{x}(\alpha+\underline{x})E\widetilde{z}}{(\alpha+\underline{z})E\widetilde{x} - (\alpha+\underline{x})E\widetilde{z}}e + (\alpha+\underline{x})R\right].
\end{aligned}$$

(b) If $E\widetilde{y} - E\widetilde{z} = \frac{\underline{z}}{\underline{x}}(E\widetilde{y} - E\widetilde{x})$ *and (SX), (SZ) do not hold then optimal choices are identical to case* (a).

Proof. Proof of (a). Since (SX) does not hold it follows that $s^* = 0$ from Lemma 6(ii). If $s^* = 0$, and (NSZ) holds then (SZ) does not hold from Lemma 10. From Lemma 8(i) all Kuhn-Tucker conditions hold if $c^1 = \frac{\underline{z}}{\alpha+\underline{z}}e$, $q = \frac{\alpha}{\alpha+\underline{z}}e$, $p = 0$, (NSZ) holds and (SZ) does not hold. This proves (i). (ii) and (iii) follow directly from Lemma 8(iii) and 9(ii).

Proof of (b). If both, (SX) and (SZ) do not hold then $s^* = 0$ by Lemma 6(iii). The result follows then from Lemma 8(i), 8(iii), 9(ii). ■

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Figure 1a − Equity Shares

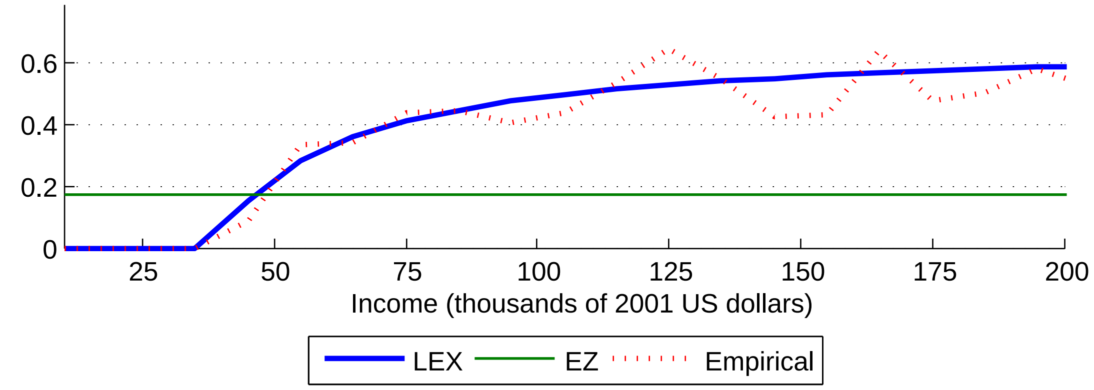

Figure 1b − Total Saving Rates

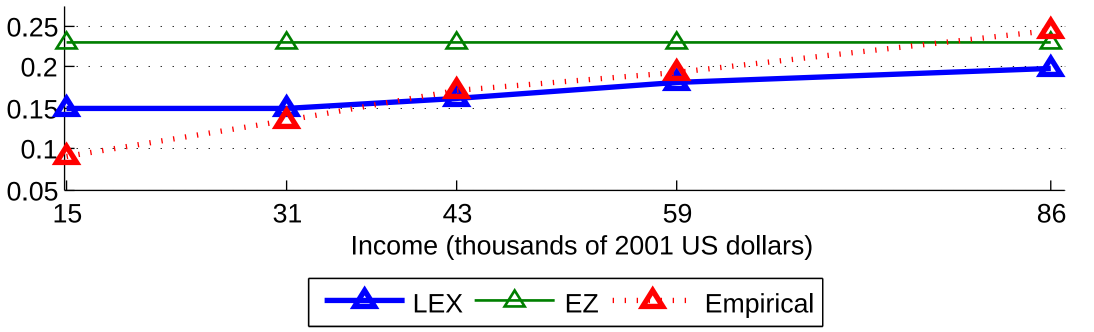

Figure 2 − Long−term downside risks of alternative pension component returns

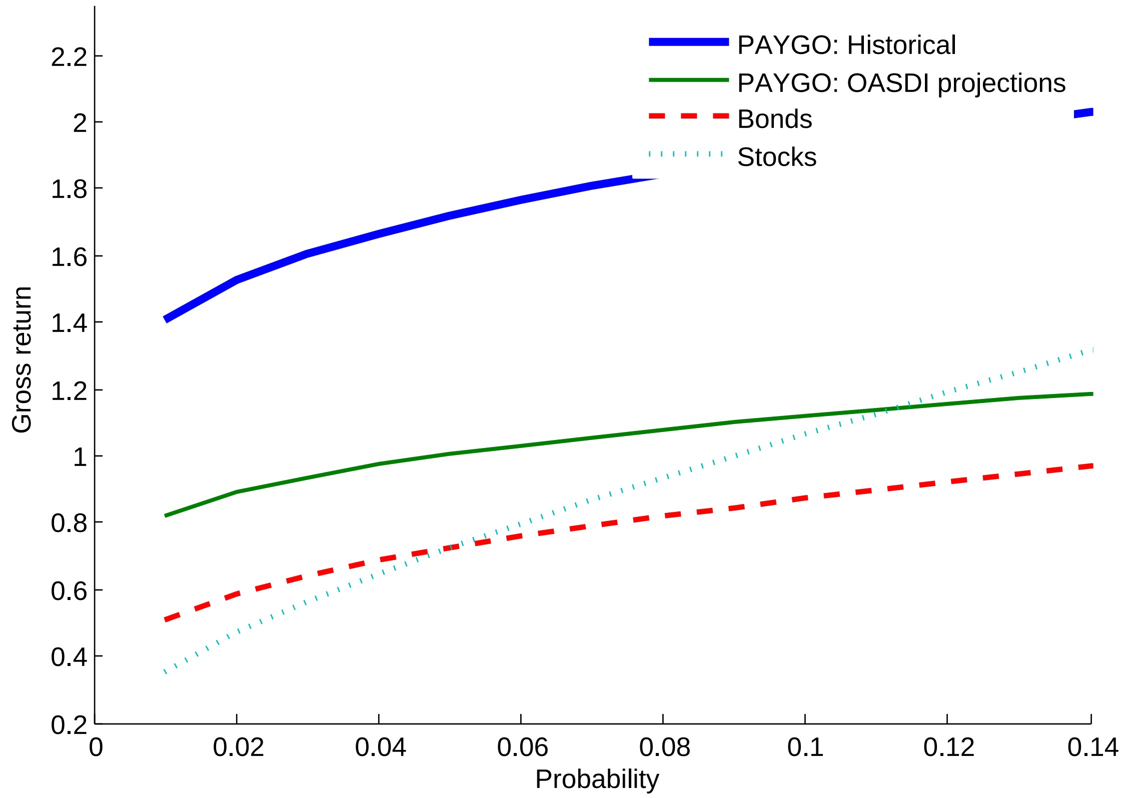

Figure 3

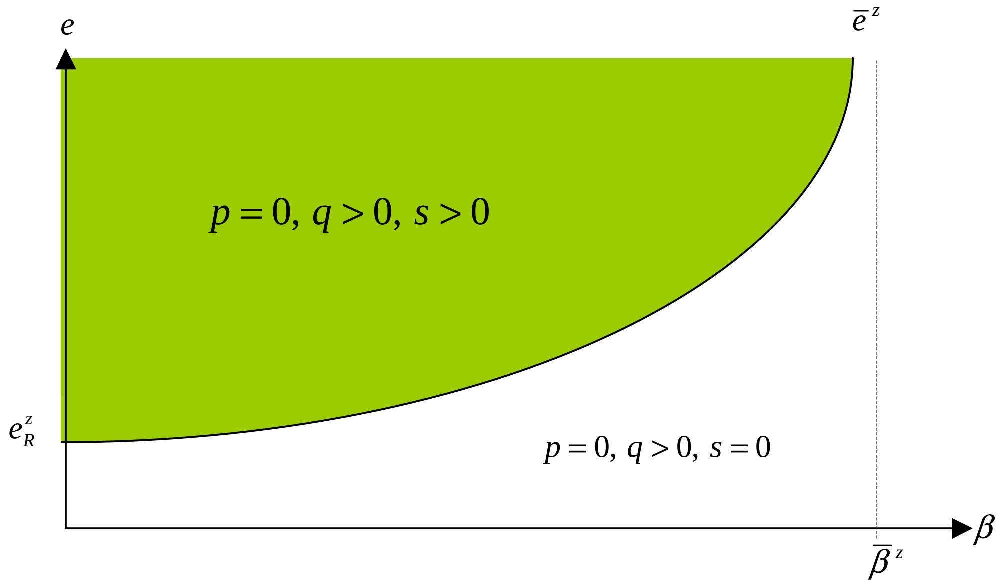

Figure 4

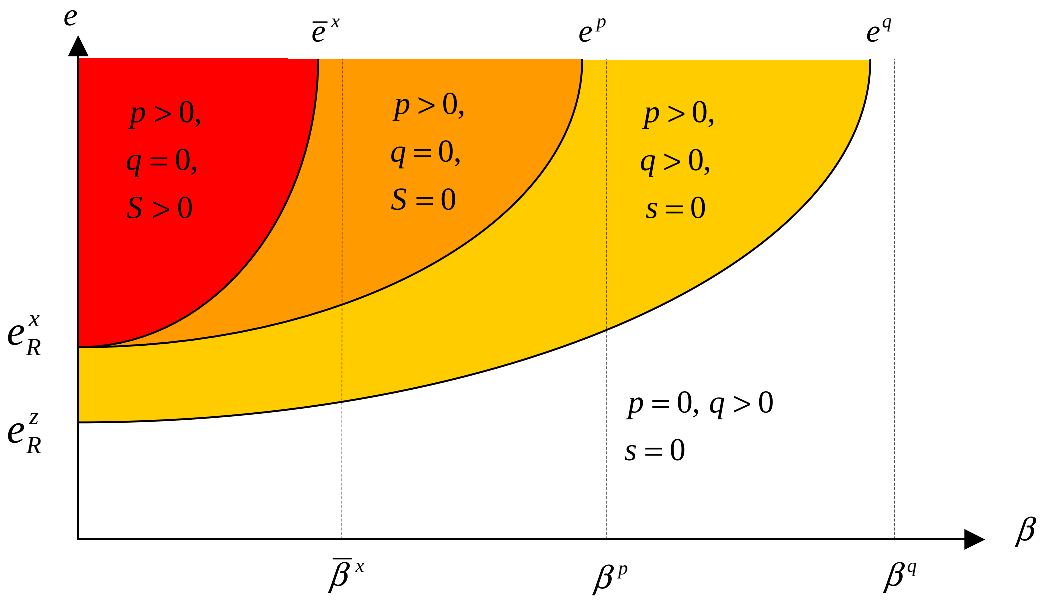
e
$\bar{e}^x$
e^p
e^q
$p > 0,$
$q = 0,$
$S > 0$
$p > 0,$
$q = 0,$
$S = 0$
$p > 0,$
$q > 0,$
$s = 0$
e_R^x
e_R^z
$p = 0, q > 0$
$s = 0$
β
$\bar{\beta}^x$
β^p
β^q

Figure 5 - $\underline{z}^{crit}$, $\underline{z}$ and $\underline{x}$ for $\underline{y} = 0$

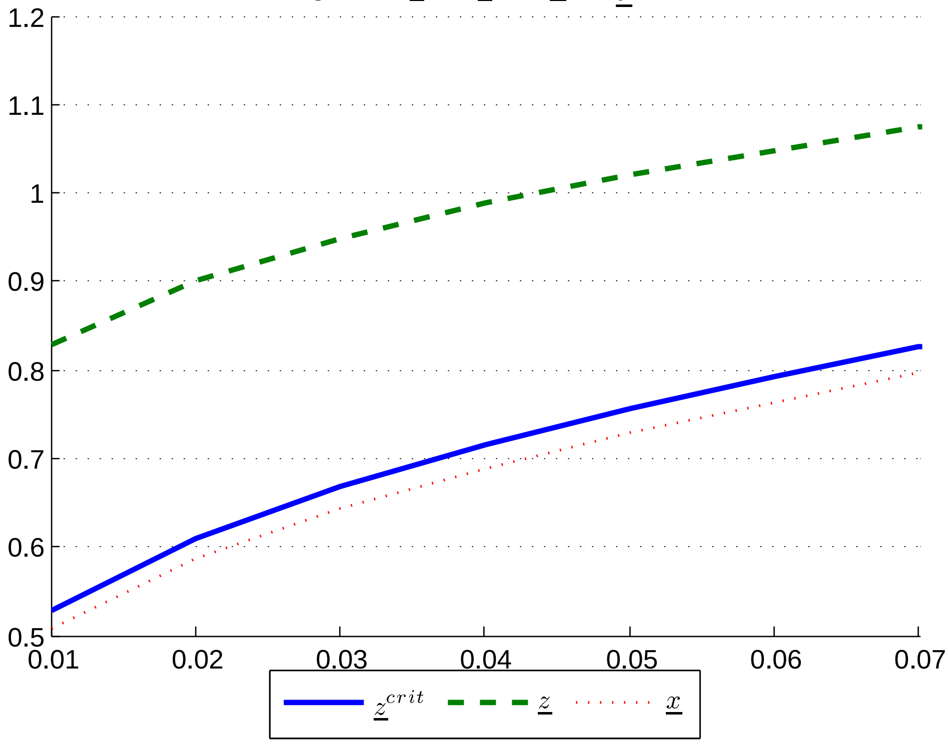

Figure 6a - Optimal PAYGO and risky saving rates, $\underline{y} = 0$

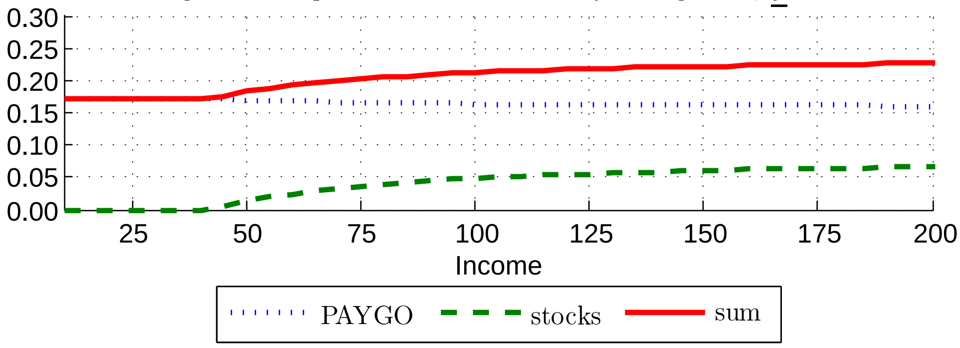

Figure 6b - Optimal PAYGO and risky saving rates, $\underline{y}$ corresponds to 3. perc.