Fit and Forecasting Performance of an Estimated Medium Run Model

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Abstract

In the past DSGE models were mainly used for policy analysis. But in recent times these models became also a favoured tool for forecasting. In this paper we evaluate the forecast performance of two different DSGE models. Until now, the fit and forecasting performance of these models wasn't the main driver of the evolution of them. With the growing interest to use DSGE models for forecasting, especially for longer time horizons, we want to know more about their forecasting performance. After estimating the models with the same data, we compare the two different DSGE models in regard to the forecasting performance and check if a medium run model outperforms a typical Business Cycle model.

JEL classification: C32, C52, E32, E37

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1. Introduction

Over the past years there was a growing interest in developing Business Cycle Models, which can reproduce Medium-Term Business Cycles. In the history of economics there were mostly two kinds of models, depending on the time pattern. On one side there are typical short-term models, exploring how an economy will perform for the next two years. And on the other side there were long-term models or so called growth models. But Medium-term projections are an important tool for economic policy. This is particularly true for national governments in the Euro Area because of the Stability and Growth Pact and the associated obligation to formulate stability programmes over five years. In addition, medium run projections are important for monetary policy because of the substantial lags of interest rate changes on inflation. Traditionally, important tools for medium run projections are relatively large-scaled macroeconomic models. But these models are estimated using single-equation methods and lack a theoretical background and especially the treatment of expectations. The new class of dynamic stochastic general equilibrium (DSGE) models are constructed for policy analysis and, over the past years, for forecasting. One reason for the advantage of forecasting with DSGE models has to do with the well-known Lucas critique and the impact of the parameters that are deep.

Until now only little effort was spent on forecasting with this type of model and evaluating its forecasting performance. But the growing literature on the forecasting performance of DSGE models indicates that these models are quite favourable over longer time horizons than time series models (e.g. Smets, Wouters 2007). However, most of these models like the New Keynesian models exhibit some aspects of economic growth but they are mainly constructed for short term horizons.

This raises the question about the features a model should have to model the medium run. This question is relatively new and there are only some answers available (Blanchard 1997, McAdam and Willman 2008, Solow 2000). Comin and Gertler (2006) show that many variables of economic activity exhibit fluctuations at medium run frequencies that are correlated with business cycle fluctuations. They also find that over the medium term output is positively correlated with technological change and R&D. This means that the in the medium run it is important to model the link between business cycle fluctuations and economic growth more closely. However, if this link is important for the medium run than a medium run model should do better forecasts for the medium term.

The aim of this paper therefore is to analyse the fit and the forecasting performance of an estimated medium run model. If the medium run is a task on its own then a medium run model should outperform business cycle models if the time horizon is extended for more than the usual rime horizon. We use the Smeets and Wouters model as a benchmark. To test whether the first generation of medium run models is already prepared for medium run forecasting we test the model by Comin and Gertler (2006) (CG-Model). In this model a two sector real business cycle model is combined with elements of endogenous technological change in the line of Romer (1990) as well as countercyclical markups and variable factor utilization.

The medium term we want to describe differs from the conventional business cycle analysis. The medium frequency we are modelling doesn't exclude the high-frequency fluctuations but builds a link between the high- and the medium-frequency business activity. So, we have one model (Smets and Wouters (2007)) (SM-Model) that tries to explain the short-term-fluctuations and one model explaining the so called medium-term cycle, including the high- and the medium-frequency.

The outline of the paper is as follows: Section 2 describes the main features of the model used for forecasting the medium-run. Section 3 presents the econometric method and estimation results. In section 4 we present some simulation results and section 5 draws some conclusions.

2. The model

We show in this section some of the main equations of the Medium-term Business Cycle model. In this model two types of final goods are produced: consumption and investment goods. For each type of goods there is an innovator of new goods and an adopter that transforms an innovation into a marketable good. We describe each sector in turn. The linearized equations of the model can be found in the Appendix.

Final goods sector

The final output composite $Y_{x,t}$ is a CES aggregate of $N_{x,t}$ final goods firms, where each firm produces a differentiated product ($Y^j_{x,t}$).

$$Y_{x,t} = \left(\int_0^{N_{x,t}} \left(Y^j_{x,t} \right)^{\frac{1}{\mu_{x,t}}} dj \right)^{\mu_{x,t}} \qquad (1)$$

Every final good producer uses several inputs: capital services ($U^j_{x,t} K^j_{x,t}$), labour ($L^j_{x,t}$), and a set of intermediate goods ($M^j_{x,t}$). The parameter $\mu_{x,t}$ >1 describes the price markup that final good producers charge. The entry/exit process will be described later. As in Comin and Gertler (2006) the technology is Cobb-Douglas, where α is the capital good share and γ is the intermediate share.

$$Y^j_{x,t} = \left[\left(U^j_{x,t} K^j_{x,t} \right)^{\alpha} \left(L^j_{x,t} \right)^{1-\alpha} \right]^{1-\gamma} \left[M^j_{x,t} \right]^{\gamma} \qquad (2)$$

$M^j_{x,t}$ itself is the intermediate goods composite and, as in Romer (1990), the source of (endogenous) productivity growth. This leads us to the intermediate goods composite.

Intermediate goods sector

The intermediate good composite is a CES aggregate, where $A_{x,t}$ is the number of the specialized goods for each sector:

$$M_{x,t}^{j} = \left(\int_{0}^{A_{x,t}} \left(M_{x,t}^{j,k} \right)^{1/\vartheta} dk \right)^{\vartheta}, \qquad (3)$$

with $\vartheta>1$. This CES formulation allows for increasing gains from expanding variety. Therefore, creation and adoption of new intermediate goods is the main source of productivity growth.

R&D and adoption

As described before, Comin and Gertler use a modified version of Romers (1990) endogenous growth model. In both sectors, the capital goods and the consumption goods sector, innovators develop new intermediate goods for the production of the final output. In a second step they sell the rights of the new created good to an adopter who uses it as employable input. Concretely, an innovator *p* in the sector *x* conducts research and development using the final consumption as input. He finances his research by borrowing from households and pays out the households who own equity stakes with the remaining profits. Let S_{xt}^{p} be the R&D expenditure by innovator *p* in sector *x* and let Z_{xt}^{p} be his stock of innovations. Furthermore, φ_{xt} is a productivity parameter and $1-\phi$ is the probability that an existing good becomes obsolete in the next period. All in all the technology process is described as follows:

$$Z_{xt+1}^{p} = \varphi_{xt} S_{xt}^{p} + \phi Z_{xt}^{p} \qquad (4)$$

We now illustrate the process how a newly developed intermediate good is adopted over time. Adopters buy the rights to the technology from the innovator and convert it into use. This process is costly as well as time-consuming. After the new product is sellable to the final goods producers they use it as an intermediate product in the production function. Let be $H_{x,t}$ the adoption expenditures, Λ_{t+1} a stochastic discount factor and let be $V_{x,t+1}$ the value to an adopter of a new product in the next period. Then the value of an “unadopted” product to the adopter follows:

$$J_{xt} = \max_{H_{xt}} \left\{ -H_{xt} + \phi E_t \left(\Lambda_{t+1} \left[\lambda_{xt} V_{xt+1} + (1-\lambda_{xt}) J_{xt+1} \right] \right) \right\} \qquad (5)$$

Through this formulation a procyclically movement in adoption expenditures can be described. It’s reasonable that in a phase of a boom the value of a marketable product will rise relative to the one still in development. So, $H_{x,t}$ will increase as it becomes more attractive to invest in adoption.

Entry/Exit and Markups

The markup, defined as the ratio of the marginal product of labor to the household's marginal rate of substitution between consumption and leisure, can be separated into a wage markup and a price markup. While the first one describes the ratio of the real wage to the household's marginal rate of substitution, the second one illustrates the ratio of the marginal product of labor to the real wage.

We characterize the coherence between the number of firms $N_{x,t}$ and the prize markup as in Galı´ and Zilibotti (1995). Following this approach, a procyclical rise in the number of firms combined with endogenous procyclical net entry induce countercyclical movements in the markup. The relation between the markup and the number of firms can be described as:

$$\mu_{x,t} = \mu(N_{x,t}) \, , \qquad (6)$$

Furthermore, each firm in sector x has to pay a per-period operating cost $b_x \psi_t$ and let be $\Pi(\mu_{x,t}, P_{x,t} Y_{x,t}^{j})$ the firm profits. So, for any firm j must hold that profits are not less than operating costs:

$$\Pi(\mu_{x,t}, P_{x,t} Y_{x,t}^{j}) \geq b_x \psi_t \qquad (7)$$

Equations (6) and (7) will determine the markup and the quantity of firms. In the general equilibrium the markup will vary countercyclically while the entry of the firms fluctuate procyclically.

Households

The household consumes, supplies labor and saves. As mentioned before the household has equity claims in the firms. A new feature is that the household has a bit of market power regarding the labor supply, which allows us to introduce wage markups. In its original form the household maximizes the present discounted utility with C_t as consumption and L_t^h as labor as follows:

$$E_t \sum_{i=0}^{\infty} \beta^i \left[ln\, C_{t+i} - \frac{\left(L_{t+i}^h\right)^{1-\zeta}}{1+\zeta} \right] \qquad (8)$$

For a better fit to the data we use the same Euler equation as in Smets and Wouters (2007) which has almost the same form but differs in one important way. While the Euler equation in Comin and Gertler depends only on the future consumption, the one in Smets and Wouters takes the past consumption into account, too. The budget constraint stays the same:

$$C_t = W_t^h L_t^h + \Pi_t + \left[D_t + P_t^k \right] K_t - P_t^k K_{t+1} + R_t B_t - B_{t+1} - T_t \, , \qquad (9)$$

where Π_t is the profit of the firms paid out to the households. Let be $W_t^h L_t^h$ the wage for the household h and D_t the dividend. The term $R_t B_t - B_{t+1}$ reflects the disposable income of the total loans and let be T_t lump-sum taxes.

As mentioned before we assume that a household has a bit of market power in labor supply, so it chooses a wage including a markup $\mu_{w,t}$ over its marginal rate of substitution between labor and consumption $\left(L_t^h\right)^{\zeta} C_t$.

$$W_t^h = \mu_{w,t} \left(L_t^h\right)^{\zeta} C_t \qquad (10)$$

Normally, the wage markup is one of the main sources to explain the high-frequency. But if it's combined with endogenous technology change and countercyclical price mark-ups as described before we find out that it's capable of explaining medium term cycles.

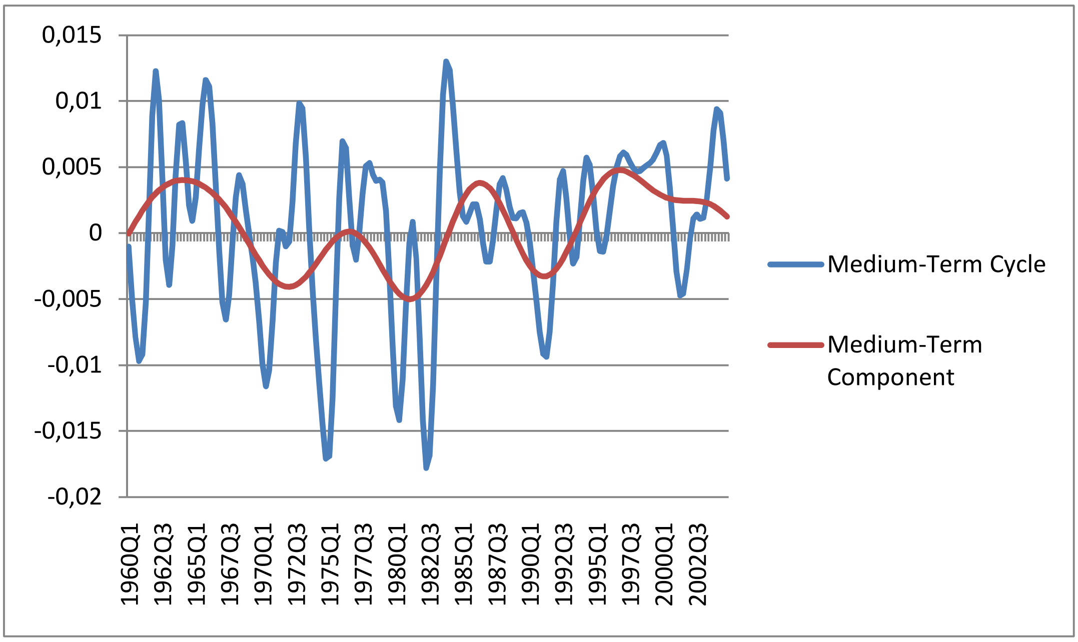
3. Calibration and Estimation

Now we present some facts about the data and the estimation of medium-terms business cycles. The data-set comprises about seven macroeconomic time-series for the SM-Model and all of them are quarterly data from 1947 to 2004. While conventional filtering identifies business cycles with frequencies between 2 and 32 quarters we use a different measurement. We define, just like Comin and Gertler (2006), the medium-term cycle existing of a high-frequency (2-32 quarters) and a medium-frequency (32-200 quarters) component. Most of the series we use are nonstationary, so we convert them into log differences and apply a band-pass-filter for the two frequencies explained above. Figure 1 plots the medium-term cycle and its medium term component. We use the same data for the CG-Model, but since the model has only four shocks we can just use four time series for the estimation and so for the forecast.

In this paper we use the Bayesian approach for parameterization and evaluating of the model, described like in An and Schorfheide (2007). An advantage of this method is the system-based analysis and the fitting of the model to a vector of aggregate time series. Furthermore, the estimation is based on a likelihood function generated by the DSGE-model instead of the discrepancy between DSGE model responses and VAR impulse responses. But the "pure" maximum likelihood estimation is not often used because of the "dilemma of absurd parameters estimates". So, the likelihood function is reweighted by a prior density.

For the estimation of a linearized DSGE model there are two algorithms that can be used to generate draws from the posterior distribution: The Random-Walk Metropolis (RWM) algorithm and the Importance Sampling (IS) algorithm. Both of them need the evaluation of the likelihood function given the prior density. In a first step the linear rational expectations system is solved to get the state transition equation. If a unique stable solution is found, then the Kalman filter is used to evaluate the likelihood function. In this paper we apply the RWM algorithm which is also implemented in the Dynare package we used for the calculations.

Figure 1
Medium-Term Business Cycle- GDP



For a comparison of the models we have to take into account few important details. Since the SM-Model was estimated for quarterly data we apply the CG-Model to quarterly data. In view of the fact that the latter one has already been applied to quarterly data (Comin and Gertler, 2009) we transfer these changes to the standard model.

First, we want to estimate the shocks of the CG-Model instead of all parameters, since we have only four shocks in the CG-Model. On one hand it's an advantage to estimate only the shocks as described in Ireland (2004) and others (DeJong et al. (2000a, b), Kim (2000), and Schorfheide (2000)) because it identifies sources of the fluctuations that are beyond the real business cycle model's technology shocks usually used. On the other hand it's a disadvantage since one has to make very clear and specific assumptions about the economy. Another reason for the estimation only of the parameters of the shock is the evidence for the most of the parameters in the literature and their appearance in other studies.

In a first analysis we assume the same values for the parameters as in Comin and Gertler (2006, 2009). After the estimation of the parameters of the shocks we apply both models to forecast at different point of times and for different time horizons.

4. Forecasting Evaluation

In this section we want to show the first results of the forecast evaluation. First we estimate both models over the same sample (1966:1-1989:4). Secondly we use the estimated models to forecast for different time horizons. Subsequently we expand the sample five times for the estimation by one year and execute the forecast again. The evaluation of the several forecasts is done by using the Root Mean Squared Error (RMSE), with four variables to be predicted: GDP, investment, consumption, and wage.

The first results of the forecast evaluation are reported in Table 1. Unfortunately the CG-Model doesn't outperform the SM-Model in the long run with 40 quarters to be predicted. The RMSE of the investment in the CG-Model is far worse than in the SM-Model. Several explanations can be responsible for the results. First, until now we only estimated the CG-Model for the different shocks but not for the parameters of the model. If we apply the estimation for all of the parameters we expect a better fit to the data. Furthermore there might be a problem in the investment equation because we introduced adjustment costs and they might affect the steady state. But some of the other variables do a better job. Especially the wage seems to be better predicted with the CG-Model, even though the forecast with 40 quarters shows a better performance in the SM-Model. In general the predictive performance of all the variables could improve by taking a closer look at the investment equation as explained before.

Table 1
Out-of-Sample Evaluation

forcast horizon	*GDP*	*INV*	*CONS*	*Wage*
		Model Smets & Wouters		
8q	0,27	1,47	0,35	0,51
20q	0,42	0,70	0,28	0,74
40q	0,57	2,12	0,43	0,62
		Model Medium-Term		
8q	0,58	9,30	0,68	0,38
20q	0,37	9,00	0,84	0,68
40q	0,80	9,83	0,75	0,80

RMSE-Root Mean Squared Error

Even though the different RMSE show no good fit to the data, for some of the variables there are also signs for the power of the model that make further investigations reasonable.

5. Conclusions

In this paper we have compared the prediction performance of two DSGE models in the case of four macroeconomic variables. While the SM-Model tries to explain the US economy in a traditional way in respect to conventional business cycles, the CG-Model uses a business cycle measurement that allows explaining medium term business cycles. With this knowledge we wanted to show the ability of the model to predict different time-series with a longer forecast horizon.

Until now first results show advantages of the SM-Model compared with the CG-Model. Especially the forecast of the investment works poorly. An improvement might be achieved by estimating the whole model instead of simply the shocks. But since only four shocks are applied we can use only four time series for the estimation of the model. But it would be reasonable to involve more time series because of the size of the CG-Model and the amount of parameters included. With further improvements to the CG-Model we expect a better forecast and further possible applications concerning the medium run.

References

An, Sungabe and Schorfheide, Frank (2007), Bayesian Analysis of DSGE Models, Econometric Reviews 26(2-4): 113-172.

Blanchard, Olivier J. (1997), The Medium Run, Brookings Papers on Economic Activity 2: 89-141.

Braun, R. Anton, Toshihiro Okada and Nao Sudou (2006), U.S. R&D and Japanese Medium Term Cycles, Bank of Japan Working Paper No. 06-E-06.

Comin, Diego, M. Gertler and A. M. Santacreu (2009), Technology Innovation and Diffusion as Sources of Output and Asset Price Fluctuations, Harvard Business School Working Paper No. 09-134.

Comin, Diego and Mark Gertler (2006), Medium-Term Business Cycles, American Economic Review 96(3): 523-551.

DeJong, D.N., Ingram, B.F., Whiteman, C.H. (2000a), Keynesian impulses versus Solow residuals: identifying sources of business cycle fluctuations, Journal of Applied Econometrics 15: 311–329.

DeJong, D.N., Ingram, B.F., Whiteman, C.H. (2000b), A Bayesian approach to dynamic macroeconomics Journal of Econometrics 98: 203–223.

Galı´ and Zilibotti (1995), Endogenous Growth and Poverty Traps in a Cournotian Model, Annales d'economie et de statistique, *0*(37–38): 197–213.

Ireland, Peter (2004), A method for taking models to the data, Journal of Economic Dynamics & Control 28: 1205-1226.

Kim, J. (2000), Constructing and estimating a realistic optimizing model of monetary policy, Journal of Monetary Economics 45: 329–359.

McAdam, Peter and Alpo Willman (2008), Medium Run Redux: Technical Change, Factor Shares and Frictions in the Euro Area, European Central Bank Working Paper No. 915.

Romer, Paul (1990), Endogenous Technological Change, Journal of Political Economy 98(5), 71-102.

Schorfheide, F. (2000), Loss function-based evaluation of DSGE models, Journal of Applied Econometrics 15: 645–670.

Smets, Frank and Rafael Wouters (2007), Shocks and Frictions in U.S. Business Cycles: A Bayesian DSGE Approach, American Economic Review 97(3): 586-606.

Solow, Robert M. (2000), Toward a Macroeconomics of the Medium Run, Journal of Economic Perspectives 14 (1): 151-158.

Appendix 1. Linearized model

1. Resource Constraint

$$0 = Y_{netm,t} - \frac{\frac{C}{Y_c}}{ynetmcons} C_t - \frac{\frac{S_k}{Y_c}}{ynetmcons} S_{k,t} - \frac{\frac{S_c}{Y_c}}{ynetmcons} S_{c,t}$$

$$- \frac{\frac{actualhc}{Y_c}}{ynetmcons} H_{c,t} - \frac{\frac{\frac{A_c}{Z_c}}{\left(1-\frac{A_c}{Z_c}\right)} \frac{actualhc}{Y_c}}{ynetmcons} Z_{c,t} + \frac{\left(\frac{Z_c}{A_c}-1\right)^{-1} \frac{actualhc}{Y_c}}{ynetmcons} A_{c,t}$$

$$- \frac{\frac{actualhk}{Y_c}}{ynetmcons} H_{k,t} - \frac{\frac{\frac{A_k}{Z_k}}{\left(1-\frac{A_k}{Z_k}\right)} \frac{actualhk}{Y_c}}{ynetmcons} Z_{k,t} + \frac{\left(\frac{Z_k}{A_k}-1\right)^{-1} \frac{actualhk}{Y_c}}{ynetmcons} A_{k,t}$$

2. Consuption Euler equation

$$0 = C_t - C_{t+1} + R_t$$

3. Aggregate Production Function

$$0 = Y_{c,t} - \frac{1}{1-bb}\psi_{z,t} - \alpha K_{t-1} + \alpha L_t - L_{y,t} - \frac{\mu_c - 1}{1-bb} N_{c,t} + \frac{\left(bb + \mu_c \log\left(N_c\right)\right)}{1-bb}\mu_{c,t}$$
$$- \alpha U_t - \frac{bb}{1-bb}\left(\frac{1}{\vartheta} - 1\right) A_{c,t}$$

4. Demand of Capital

$$0 = Y_{c,t} - KL_t + L_t - L_{y,t} - \mu_{c,t} - \frac{1}{\left(\frac{1+\delta}{\frac{D}{P_k}}\right)} D_t - \frac{\delta}{\frac{D}{P_k}+\delta} P_{k,t} - edu \frac{1}{\frac{\frac{D}{P_k}}{\delta}+1} U_t$$

5. Capacity choice

$$0 = Y_{c,t} - \left(1 + edp\right) U_t - \mu_{c,t} - KL_t + L_t - L_{y,t} - P_{k,t}$$

6. Labor demand in final output

$$0 = Y_t - \mu_{c,t} - L_{y,t} - W_t$$

7. Production of new investment goods

$$0 = Y_{k,t} - \frac{1}{1-bb}\psi_t - \alpha\, KL_t - \frac{\alpha - 1}{1-\frac{L_c}{L}} L_t - \alpha U_t + \frac{\frac{L_c}{L}}{1-\frac{L_c}{L}} L_{y,t} - \frac{\mu_k - 1}{1-bb} N_{k,t}$$

$$+ \frac{\left(bb + \mu_k \log\left(N_k\right)\right)}{1-bb}\mu_{k,t} - \frac{bb}{1-bb} P_{k,t} - \frac{bb}{1-bb}\left(\frac{1}{\vartheta} - 1\right) A_{k,t}$$

8. Real wage

$$0 = W_t - \zeta L_t - C_t - \mu_{w,t}$$

9. Profits embodied

$$0 = \Pi_{k,t} - Y_{k,t} - P_{k,t} + \mu_{k,t}$$

10. Profits disembodied

$$0 = \Pi_{c,t} - Y_{c,t} + \mu_{c,t}$$

11. Value of an adopted innovation for embodied

$$0 = V_{k,t} - \left(1 - \frac{\Pi_{m,k}}{V_k}\right) A_{k,t} - \left(\frac{\Pi_{m,k}}{V_k}\right)\Pi_{m,k,t} - \left(1 - \frac{\Pi_{m,k}}{V_k}\right) V_{k,t+1} + \left(1 - \frac{\Pi_{m,k}}{V_k}\right) A_{k,t+1}$$

$$+ \left(1 - \frac{\Pi_{m,k}}{V_k}\right) R_t$$

12. Value of an adopted innovation for disembodied

$$0 = V_{c,t} - \left(1 - \frac{\Pi_{m,c}}{V_c}\right) A_{c,t} - \left(\frac{\Pi_{m,c}}{V_c}\right)\Pi_{m,c,t} - \left(1 - \frac{\Pi_{m,c}}{V_c}\right) V_{c,t+1} \left(1 - \frac{\Pi_{m,c}}{V_c}\right) A_{c,t+1}$$

$$+ \left(1 - \frac{\Pi_{m,c}}{V_c}\right) R_t$$

13. Capital accumulation

$$0 = K_t + \frac{\left(edu\, \delta\right)}{\left(1 + g_k\right)} u_t - jcof\; K_{t-1} - \left[1 - jcof\,\right] Y_{k,t}$$

with

$$jocof = \frac{\left(1 - \delta\right)}{\left(1 + g_k\right)}$$

14. Law of motion for embodied productivity

$$0 = Z_{k,t} - Z_{k,t-1} - \rho \frac{\left(g_{zk} + ok\right)}{\left(1 + g_{zk}\right)} S_{k,t-1} + \rho \frac{\left(g_{zk} + ok\right)}{\left(1 + g_{zk}\right)} \Psi_{t-1} - \frac{\left(g_{zk} + ok\right)}{\left(1 + g_{zk}\right)} \chi_{k,t-1}$$

with $ok=(1-\phi)$

15. Law of motion for disembodied productivity

$$0=Z_{c,t}-Z_{c,t-1}-\rho\frac{(g_{zc}+oc)}{(1+g_{zc})}S_{c,t-1}+\rho\frac{(g_{zc}+oc)}{(1+g_{zc})}\Psi_{t-1}$$

16. Free entry for embodied

$$0=(1-\rho)S_{k,t}-Z_{k,t}+\rho\Psi_{t}-J_{k,t+1}+Z_{k,t+1}+R_{t}$$

17. Free entry for disembodied

$$0=(1-\rho)S_{c,t}-Z_{c,t}+\rho\Psi_{t}-J_{c,t+1}+Z_{c,t+1}+R_{t}$$

18. Bellman for not adopted disembodied innovation

$$\begin{aligned}0=&-J_{c,t}-\left[\frac{H_c}{J_c}+\phi_c\,elc\,\lambda_c\frac{1}{R}rz\left(1-\frac{Z_c}{A_c}\frac{V_c}{J_c}\right)\right]H_{c,t}-\left(1+\frac{H_c}{J_c}\right)R_t\\&+\phi_c rz\frac{1}{R}\left((1-\lambda_c)+\lambda_c\frac{Z_c}{A_c}\frac{V_c}{J_c}\right)Z_{c,t}-\phi_c\lambda_c\frac{1}{R}rz\frac{Z_c}{A_c}\frac{V_c}{J_c}A_{c,t+1}+\phi_c\lambda_c\frac{1}{R}rz\frac{Z_c}{A_c}\frac{V_c}{J_c}V_{c,t+1}\\&-\phi_c\,elc\,\lambda_c\frac{1}{R}rz\left(\frac{Z_c}{A_c}\frac{V_c}{J_c}-1\right)\Psi_t-\phi_c\frac{1}{R}rz(1-\lambda_c)Z_{c,t+1}+\phi_c\frac{1}{R}rz(1-\lambda_c)J_{c,t+1}\end{aligned}$$

19. Law of motion for adopted disembodied innovation

$$\begin{aligned}0=&A_{c,t}-\phi_c\frac{(1-\lambda_c)}{1+g_{zc}}A_{c,t-1}-elc\,\lambda_c\left(\frac{\phi_c}{1+g_{zc}}\frac{Z_c}{A_c}-\frac{\phi_c}{1+g_{zc}}\right)H_{c,t-1}\\&+elc\,\lambda_c\left(\frac{\phi_c}{1+g_{zc}}\frac{Z_c}{A_c}-\frac{\phi_c}{1+g_{zc}}\right)\Psi_{t-1}-\left(1-(1-\lambda_c)\frac{\phi_c}{1+g_{zc}}\right)Z_{c,t-1}\end{aligned}$$

20. Optimal investment in adoption of disembodied innovation

$$\begin{aligned}0=&Z_{c,t}-(1+ellc)\Psi_t-R_t+ellc\,H_{c,t}-\frac{1}{1-\frac{J_c}{V_c}\frac{A_c}{Z_c}}V_{c,t+1}-\frac{1}{1-\frac{J_c}{V_c}\frac{A_c}{Z_c}}A_{c,t+1}\\&-\frac{1}{\frac{V_c}{J_c}\frac{Z_c}{A_c}-1}J_{c,t+1}+\frac{1}{\frac{V_c}{J_c}\frac{Z_c}{A_c}-1}Z_{c,t+1}\end{aligned}$$

21. Bellman for not adopted embodied innovation

$$0 = -J_{k,t} - \left[\frac{H_k}{J_k} + \phi_k \, elk \, \lambda_k \frac{1}{R} ra \left(1 - \frac{Z_k}{A_k} \frac{V_k}{J_k} \right) \right] H_{k,t} - \left(1 + \frac{H_k}{J_k} \right) R_t$$

$$+ \phi_k ra \frac{1}{R} \left((1-\lambda_k) + \lambda_k \frac{Z_k}{A_k} \frac{V_k}{J_k} \right) Z_{k,t} - \phi_k \lambda_k \frac{1}{R} ra \frac{Z_k}{A_k} \frac{V_k}{J_k} A_{k,t+1} + \phi_k \lambda_k \frac{1}{R} ra \frac{Z_k}{A_k} \frac{V_k}{J_k} V_{k,t+1}$$

$$- \phi_k \, elk \, \lambda_k \frac{1}{R} ra \left(\frac{Z_k}{A_k} \frac{V_k}{J_k} - 1 \right) \Psi_t - \phi_k \frac{1}{R} ra (1-\lambda_k) Z_{k,t+1} + \phi_k \frac{1}{R} ra (1-\lambda_k) J_{k,t+1}$$

22. Law of motion for adopted embodied innovation

$$0 = A_{k,t} - \phi_k \frac{(1-\lambda_k)}{1+g_{zk}} A_{k,t-1} - elk \, \lambda_k \left(\frac{\phi_k}{1+g_{zk}} \frac{Z_k}{A_k} - \frac{\phi_k}{1+g_{zk}} \right) H_{k,t-1}$$

$$+ elk \, \lambda_k \left(\frac{\phi_k}{1+g_{zk}} \frac{Z_k}{A_k} - \frac{\phi_k}{1+g_{zk}} \right) \Psi_{t-1} - \left(1 - (1-\lambda_k) \frac{\phi_k}{1+g_{zk}} \right) Z_{k,t-1}$$

23. Optimal investment in adoption of embodied innovation

$$0 = Z_{k,t} - (1+ellk)\Psi_t - R_t + ellk \, H_{k,t} + \frac{1}{1 - \frac{J_k}{V_k} \frac{A_k}{Z_k}} V_{k,t+1} - \frac{1}{1 - \frac{J_k}{V_k} \frac{A_k}{Z_k}} A_{k,t+1}$$

$$- \frac{1}{\frac{V_k}{J_k} \frac{Z_k}{A_k} - 1} J_{k,t+1} + \frac{1}{\frac{V_k}{J_k} \frac{Z_k}{A_k} - 1} Z_{k,t+1}$$

24. Arbitrage

$$0 = P_{k,t} + R_t - \frac{(R-1-g_{pk})}{R} D_{t+1} - \frac{(1+g_{pk})}{R} P_{k,t+1}$$

25. Entry into final goods sector

$$0 = \mu_{c,t} + mucof \, Y_{c,t} - mucof \, \Psi_t - mucof \, N_{c,t}$$

26. Markup in consuption goods sector

$$0 = \mu_{c,t} - etamu \, N_{c,t}$$

27. Entry into capital goods sector

$$0 = \mu_{k,t} + mukcof \, Y_{k,t} + mukcof \, P_{k,t} - mukcof \, \Psi_t - mukcof \, N_{k,t}$$

28. Markup in investment goods sector

$$0 = \mu_{k,t} - etamk \, N_{k,t}$$

29. Definition of output net of total overhead costs

$$0 = Y_{net,t} - \frac{1}{1-\frac{oc}{Y_c}} Y_{c,t} + \frac{\frac{occ}{Y_c}}{1-\frac{oc}{Y_c}} N_{c,t} + \frac{\frac{ock}{Y_c}}{1-\frac{oc}{Y_c}} N_{k,t} + \frac{\frac{oc}{Y_c}}{1-\frac{oc}{Y_c}} \Psi_t$$

30. Definition of scaling factor

$$0 = \Psi_t - K_{t-1} - \gamma\left(1-\frac{1}{\vartheta}\right) A_{k,t} + \gamma\left(1-\frac{1}{\vartheta}\right) A_{c,t}$$

31. Definition of ynetm

$$0 = Y_{netm,t} - \frac{1}{1-\frac{M_c}{Y_c}\left(\frac{Y_{net}}{Y_c}\right)^{-1} - \frac{M_k}{Y_c}\left(\frac{Y_{net}}{Y_c}\right)^{-1}} Y_{net,t} + \frac{\frac{M_c}{Y_c}}{ynetmcons} Y_{c,t} - \frac{\frac{M_c}{Y_c}}{ynetmcons} \mu_{c,t}$$
$$+ \frac{\frac{M_k}{Y_c}}{ynetmcons} P_{k,t} + \frac{\frac{M_k}{Y_c}}{ynetmcons} Y_{k,t} - \frac{\frac{M_k}{Y_c}}{ynetmcons} \mu_{k,t}$$

32. Definition of total value added

$$0 = Y_{tot,t} - \frac{ynetmcons}{ynetmcons + \frac{P_k Y_k}{Y_c}} Y_{netm,t} - \frac{\frac{P_k Y_k}{Y_c}}{ynetmcons + \frac{P_k Y_k}{Y_c}} P_{k,t} - \frac{\frac{P_k Y_k}{Y_c}}{ynetmcons + \frac{P_k Y_k}{Y_c}} Y_{k,t}$$

33. Labor demand in capital goods production

$$0 = Y_{k,t} + P_{k,t} - \mu_{k,t} - W_t - \frac{1}{1-\frac{L_c}{L}} L_t - \frac{\frac{L_c}{L}}{1-\frac{L_c}{L}} L_{y,t}$$

34. Embodied productivity shock process

$$\chi_t = \rho_\chi \chi_{t-1} + \varepsilon_{\chi,t}$$

35. Labor augmenting technology shock process

$$\chi_{z,t} = \rho_{\chi_z} \chi_{z,t-1} + \varepsilon_{\chi_z,t}$$

36. Wage markup shock process

$$\mu_{w,t} = \rho_{\mu_w} \mu_{w,t-1} + \varepsilon_{\mu_w,t}$$

37. Disembodied productivity shock process

$$\chi_{k,t} = \rho_{\chi_k} \chi_{k,t-1} + \varepsilon_{\chi_k,t}$$