Topics in Experimental and Behavioral Economics

Quantal Response - A Noisy Equilibrium

Slide 1

Arno Riedl
Maastricht University

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Introduction

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1. Quantal Response Equilibrium
2. Logit equilibrium
3. Some experiments
4. Logit equilibrium in continuous strategies
5. Examples and applications

Quantal Response Equilibrium

McKelvey & Palfrey GEB95

Slide 3

- Consider a **Normal Form Game** $\Gamma = \{N, S, u\}$
- n players: $N = \{1, ..., n\}$
  strategy set $S_i = \{s_{i1}, ..., s_{iJ_i}\}$, J_i pure strategies of player i
  payoff function $u_i : S \to \mathcal{R}$, where $S = \Pi_{i \in N} S_i$
- J_i-dim simplex $\Delta_i = \{p_i = (p_{i1}, ..., p_{iJ_i}) : \sum p_i = 1, p_{ij} \geq 0\}$;
  p_i is a mixed strategy for player i
- $\Delta = \Pi_{i \in N} \Delta_i$ space of mixed strategies;
  $p = (p_1, ..., p_n)$ vector of mixed strategies
- payoff $u_i(p) = \sum_{s \in S} p(s) u_i(s)$, where $p(s) = \Pi_{i \in N} p_i(s_i)$
  (expected payoff when players play mixed strategies p

QRE - Definitions

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- Definition of **Nash Equilibrium**:
  A vector $p = (p_1, ..., p_n) \in \Delta$ of (mixed) strategies is a *Nash equilibrium* if for all $i \in N$ and all $p'_i \in \Delta_i$, $u_i(p'_i, p_{-i}) \leq u_i(p)$

QRE - Definitions

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- Random utility - Utility with noise
  - space of possible payoffs of player i:
    $X_i = R^{J_i}$; $X = \Pi_{i=1}^n X_i$
  - Define $\bar{u} : \Delta \to X$ as

$$\bar{u}(p) = (\bar{u}_1(p), ..., \bar{u}_n(p)) \text{ where } \bar{u}_{ij}(p) = u_i(s_{ij}, p_{-i})$$

  - player i's utility is subject to random error:

$$\hat{u}_{ij}(p) = \bar{u}_{ij}(p) + \epsilon_{ij}$$

    where $\epsilon_i = (\epsilon_{i1}, ..., \epsilon_{iJ_i})$ player i's error vector, with $E[\epsilon_i] = 0$ and density function $f_i(\epsilon_i)$.

QRE - Definitions

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- **Behavioral assumption**:
  - each player i selects an action j such that $\hat{u}_{ij} \geq \hat{u}_{ik}$ for all $k = 1, ..., J_i$
- Define ij-response set $R_{ij} \subseteq R^{J_i}$ as

$$R_{ij}(\bar{u}_i) = \{\epsilon_i \in R^{J_i} | \bar{u}_{ij} + \epsilon_{ij} \geq \bar{u}_{ik} + \epsilon_{ik} \text{ for all } k = 1, ..., J_i\}$$

  (region of errors that will lead i to choose j, for given p)
- **Statistical response function** gives the probability that player i will choose action j, given $\bar{u}$ and f

$$\sigma_{ij}(\bar{u}_i) = \int_{R_{ij}(\bar{u}_i)} f(\epsilon) d\epsilon$$

QRE - Definitions

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- Definition of **Quantal Response Equilibrium**:

  For any admissible $f = (f_1, ..., f_n)$ and game $\Gamma = (N, S, u)$ a *quantal response equilibrium* (QRE) is any vector $\pi \in \Delta$ such that for all $i \in N$ and all $1 \leq j \leq J_i$, $\pi_{ij} = \sigma_{ij}(\bar{u}_i(\pi))$.

  $\sigma_i : R^{J_i} \rightarrow \Delta^{J_i}$ is the *statistical reaction function* (or *quantal response function*) of player i.

- QRE is a statistical version of NE, where each agent's expected utility is subject to a random error, e.g. due to calculation error.

QRE - Properties

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- **Properties** of the **statistical reaction function**:
  1. $\sigma \in \Delta$ is non-empty.
  2. σ_i is continuous on R^{J_i}.
  3. σ_{ij} monotonically increasing in $\bar{u}_{ij}$.
  4. $\bar{u}_{ij} > \bar{u}_{ik} \Rightarrow \sigma_{ij}(\bar{u}) > \sigma_{ik}(\bar{u})$.

- 1. and 2. imply **existence of QRE**, for any admissible f.
- 3. and 4. imply 'better actions are more likely to be chosen'.

Logit Equilibrium

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- A simple parametric class of QRE - **Logit Equilibrium**: If errors are independent with extreme value distribution, then for payoff $x_i \in \mathcal{R}^{J_i}$ the **logistic** quantal response function is given by

$$\sigma_{ij}(x_i) = \frac{e^{\lambda x_{ij}}}{\sum_{k=1}^{J_i} e^{\lambda x_{ik}}}$$

- QRE or **Logit Equilibrium** requires

$$\pi_{ij} = \frac{e^{\lambda x_{ij}}}{\sum_{k=1}^{J_i} e^{\lambda x_{ik}}}, \qquad \text{where } x_{ij} = \bar{u}_{ij}(\pi)$$

parameter λ inversely related to level of error:
$\lambda = 0$: all error; all probabilities equal $1/J_i$
$\lambda = +\infty$: no error; Logit Eq. approaches NE

Logit Equilibrium

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- **Properties** of the **Logit Equilibrium**:
- Define the **Logit Equilibrium correspondence** $\pi^* : \mathcal{R}^+ \to 2^\Delta$ by

$$\pi^*(\lambda) = \left\{ \pi \in \Delta : \pi_{ij} = \frac{e^{\lambda \bar{u}_{ij}(\pi)}}{\sum_{k=1}^{J_i} e^{\lambda \bar{u}_{ik}(\pi)}} \ \forall i, j \right\}$$

- Note: For $\lambda = 0$

$$\pi^*(0) = \left\{ \pi \in \Delta : \pi_{ij} = \frac{e^0}{\sum_{k=1}^{J_i} e^0} \ \forall i, j \right\} = (1/J_1, 1/J_2, ..., 1/J_n)$$

Logit Equilibrium

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- **Theorem**: Let σ be the logistic quantal response function, and $\{\lambda_1, \lambda_2, ...\}$ be a sequence s.th. $\lim_{t\to\infty} \lambda_t = \infty$. Let $\{p_1, p_2, ...\}$ be a corresponding sequence with $p_t \in \pi^*(\lambda_t)\ \forall t$, s.th. $\lim_{t\to\infty} p_t = p^*$, then p^* is a Nash equilibrium.
- **Theorem**: For almost all games $\Gamma = (N, S, u)$
  1. $\pi^*(\lambda)$ is odd for almost all λ
  2. π^* is upper hemicontinuous
  3. The graph of π^* contains a unique branch from the centroid for $\lambda = 0$ to a unique NE as λ goes to $+\infty$

Logit Equilibrium and experiments

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- Some first laboratory testing of the Logit Equilibrium for two-person games
  with a unique Nash Equilibrium
  and without payoffs Pareto preferred to the NE
- Idea: Calculate maximum likelihood estimate of λ and check how well it fits the data.
- Analyzed experiments span more than 30 years $\rightarrow$ all payoffs converted to real 1982 dollars

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Logit Equilibrium and experiments

- Laboratory Experiment I (Lieberman, 1960) - Two person zero sum game:

	B_1	B_2	B_3
A_1	15,-15	0,0	-2,2
A_2	0,0	-15,15	-1,1
A_3	1,-1	2,-2	0,0

- unique Nash equilibrium (A_3, B_3)
- each subject in 200 plays with single opponent

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Logit Equilibrium and experiments

	B_1	B_2	B_3
A_1	15,-15	0,0	-2,2
A_2	0,0	-15,15	-1,1
A_3	1,-1	2,-2	0,0

- Lieberman zero sum game:

TABLE III
DATA AND ESTIMATES FOR LIEBERMAN (1960) EXPERIMENTS: $N = 300$ FOR EACH EXPERIENCE LEVEL

	Actual Data				Predicted			
Periods	A_1	A_3	B_2	B_3	$\hat{A}_1$ $\hat{B}_2$	$\hat{A}_3$ $\hat{B}_3$	λ	$-\mathcal{L}^*$
1–10	0.260	0.720	0.300	0.667	0.277	0.696	0.176	212.0
11–20	0.167	0.806	0.227	0.760	0.196	0.781	0.252	177.0
21–30	0.113	0.880	0.160	0.833	0.138	0.838	0.329	134.3
31–40	0.093	0.887	0.120	0.853	0.106	0.869	0.390	134.4
41–50	0.060	0.907	0.073	0.907	0.066	0.906	0.500	109.5
51–60	0.060	0.873	0.120	0.860	0.087	0.886	0.435	144.7
61–70	0.060	0.853	0.113	0.867	0.083	0.890	0.448	152.7
71–80	0.060	0.907	0.047	0.933	0.054	0.916	0.547	98.9
81–90	0.047	0.893	0.067	0.920	0.056	0.915	0.542	112.3
91–100	0.027	0.920	0.080	0.907	0.053	0.918	0.553	105.6
101–120	0.053	0.907	0.047	0.933	0.051	0.920	0.564	99.5
111–120	0.027	0.920	0.047	0.933	0.037	0.932	0.635	94.2
121–130	0.040	0.927	0.040	0.920	0.040	0.929	0.616	97.1
131–140	0.033	0.927	0.047	0.953	0.040	0.929	0.616	80.2
141–150	0.053	0.913	0.060	0.900	0.056	0.915	0.542	112.3
151–160	0.053	0.900	0.053	0.920	0.052	0.919	0.558	109.3
161–170	0.027	0.946	0.060	0.927	0.045	0.925	0.592	83.4
171–180	0.053	0.900	0.033	0.927	0.042	0.927	0.604	107.1
181–190	0.027	0.933	0.020	0.973	0.023	0.946	0.737	67.0
191–200	0.040	0.920	0.047	0.933	0.044	0.926	0.598	93.7

- data broken down to 20 experience levels of 10 periods QRE estimates for each level
- during early rounds player 1 overplays A_1 and player 2 overplays B_2, relative to NE; both predicted by QRE (see fig. next slide)

Logit Equilibrium and experiments

- Lieberman zero sum game:

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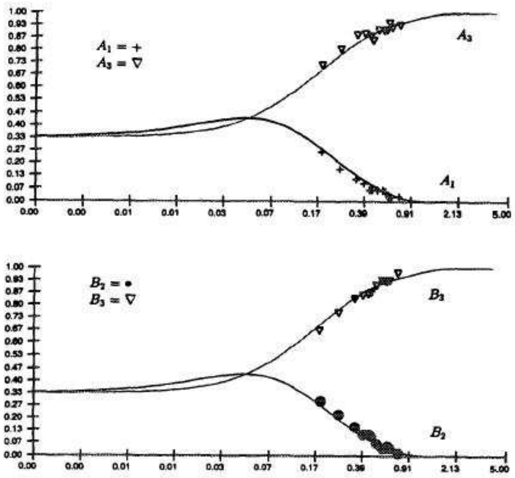

FIG. 3. QRE as a function of λ for Lieberman experiment.

- QRE prediction and actual data

Logit Equilibrium and experiments

- Laboratory Experiment II (O'Neill, 1987) - Zero sum game

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	B_1	B_2	B_3	B_4
A_1	5,-5	-5,5	-5,5	-5,5
A_2	-5,5	-5,5	5,-5	5,-5
A_3	-5,5	5,-5	-5,5	5,-5
A_4	-5,5	5,-5	5,-5	-5,5

- unique Nash equilibrium in mixed strategies: $(0.4, 0.2, 0.2, 0.2)$ both players
- each subject in 105 plays

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Logit Equilibrium and experiments

- O'Neill zero sum game:

TABLE IV
DATA AND ESTIMATES FOR O'NEILL

	Number	Frequency	Rand	NE	QRE
A_1	949	0.362	0.250	0.400	0.360
A_2	579	0.221	0.250	0.200	0.213
A_3	565	0.215	0.250	0.200	0.213
A_4	532	0.203	0.250	0.200	0.213
B_1	1119	0.426	0.250	0.400	0.426
B_2	592	0.226	0.250	0.200	0.191
B_3	470	0.179	0.250	0.200	0.191
B_4	444	0.169	0.250	0.200	0.191
λ			0	∞	1.313
$-\mathcal{L}^*$			7278	7016	7004

- data broken down to 7 experience levels of 15 periods
- data show: A_1 is 'underplayed'; B_1 is 'overplayed'
- QRE predicts: $A_1 < B_1$ is 'overplayed'
- QRE does significantly better than Nash and Random model; LR test

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Logit Equilibrium and experiments

- O'Neill zero sum game:

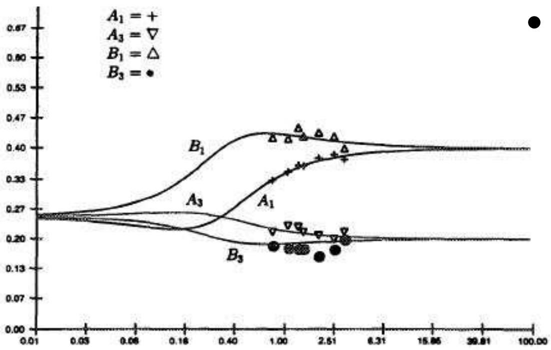

FIG. 4. QRE as a function of λ for O'Neill experiment.

- QRE prediction:

  for intermediate λ: B_1 overplayed, A_1 underplayed

  for all λ: $B_1 > A_1$

Logit Equilibrium and experiments

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- Laboratory Experiment IV (Ochs, 1995) - (Non-)zero-sum games

Game 1	B_1	B_2
A_1	1,0	0,1
A_2	0,1	1,0

Game 2	B_1	B_2
A_1	9,0	0,1
A_2	0,1	1,0

Game 3	B_1	B_2
A_1	4,0	0,1
A_2	0,1	1,0

- Only difference is payoff to player 1 if (A_1, B_1) is played.
- In the unique Nash equilibrium:
  player 1 mixes always with 0.5
  player 2 plays B_1 with probability 0.5, 0.1, and 0.2, respectively.

Logit Equilibrium and experiments

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- Ochs 95 - Non-zero-sum game:
  52 plays, 4 experience levels of 16 periods

TABLE IX
DATA AND ESTIMATES FOR OCHS. GAME 2

Period	n	Actual		Predicted		λ	$-\mathfrak{L}^*$		
		A_1	B_1	$\hat{A}_1$	$\hat{B}_1$		QRE	Nash	Rand
1–16	128	0.541	0.326	0.645	0.347	1.951	1721	1938	1774
17–32	128	0.649	0.228	0.645	0.228	3.763	1517	1664	1774
33–48	128	0.578	0.250	0.648	0.241	3.475	1605	1725	1774
48–52	64	0.626	0.200	0.636	0.197	4.638	743	792	887
All	448	0.595	0.258	0.649	0.254	3.241	5612	6119	6210

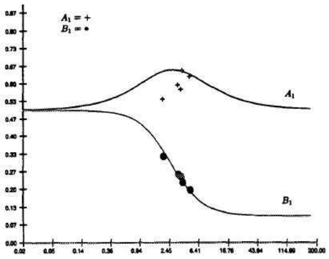

FIG. 6. QRE as a function of λ for Game 2 of the Ochs experiment.

- Game 2: both A_1 and B_1 are overplayed (Nash: 0.5 and 0.1)
  QRE: for intermediate λ precisely this is predicted
  Note: λ increases over time $\rightarrow$ suggests learning

Logit Equilibrium and experiments

- Ochs 95 - Non-zero-sum game:
  52 plays, 4 experience levels of 16 periods

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TABLE X
DATA AND ESTIMATES FOR OCHS, GAME 3

		Actual		Predicted			$-\mathfrak{L}^*$		
Period	n	A_1	B_1	$\hat{A}_1$	$\hat{B}_1$	λ	QRE	Nash	Rand
1–16	128	0.527	0.366	0.615	0.383	1.856	1747	1822	1774
17–32	128	0.573	0.393	0.610	0.405	1.568	1735	1870	1774
33–48	128	0.610	0.302	0.614	0.301	3.306	1640	1708	1774
48–52	128	0.455	0.285	0.500	0.200	∞	1679	1679	1774
All	512	0.542	0.336	0.619	0.331	2.656	6864	7079	7098

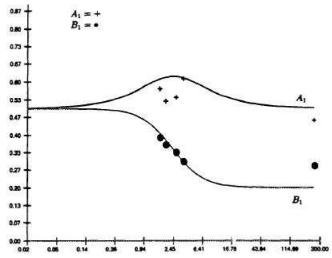

FIG. 7. QRE as a function of λ for Game 3 of the Ochs experiment.

- Game 3: both A_1 and B_1 are overplayed (Nash: 0.5 and 0.2)
  QRE: for intermediate λ precisely this is predicted
  Note: λ increases over time → suggests learning

Logit equilibrium - Continuous strategies

Anderson, Goeree, & Holt, SEJ 2002

- Conceptual issues
- Theoretical results

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- Applications:
  - The travelers dilemma
  - Minimal effort game
  - Public goods game

Conceptional issues

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- As in the original approach of MP95:
  - If decision x has expected payoff $\pi_i^e(x)$ than a player i chooses the decision with the highest value for $U_i(x) = \pi_i^e(x) + \mu\epsilon_{ix}$.
  - μ is a positive 'error' parameter and ϵ_{ix} is the realization of a random variable
  - <u>Result</u>: choice is stochastic and distribution of the random variable determines the form of the choice probabilities.
  - Double exponential distribution (extreme values) $\rightarrow$ Logit model

Conceptional issues - choice density

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- <u>Logit model</u>: choice probabilities are proportional to exponential functions of expected payoffs
- In particular, $prob_i(x)$ is proportional to $e^{\pi_i^e(x)/\mu}$
  Note: Higher μ makes choice probabilities less sensitive to expected payoff (compare with $1/\lambda$ in MP95)
- Continuum of choice possibilities $x \in [\underline{x}, \bar{x}]$ then Logit model specifies the following choice density:

$$f_i(x) = \frac{e^{\pi_i^e(x)/\mu}}{\int_{\underline{x}}^{\bar{x}} e^{\pi_i^e(y)/\mu} dy}, \ \forall i \tag{0.1}$$

- Note: this is a 'continuous' version of the logistic quantal response function in MP95.

Conceptional issues - equilibrium

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- Application of choice models to games: take into account that distributions of <u>other's</u> decisions enter the payoff function in choice density function (**??**)
- Hence, consistency requirement as for Nash equilibrium: belief distributions that determine expected payoffs (r.h.s.) match the decision distributions (l.h.s)
- Thus, the logit choice rule (**??**) determines the equilibrium distributions as a fixed point → the QRE in Logit form known from MP95.

$$f_i(x) = \frac{e^{\pi_i^e(x)/\mu}}{\int_{\underline{x}}^{\bar{x}} e^{\pi_i^e(y)/\mu} dy}, \ \forall x \ \forall i \tag{0.2}$$

Conceptional issues - equilibrium density

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$$\boxed{f_i(x) = \frac{e^{\pi_i^e(x)/\mu}}{\int_{\underline{x}}^{\bar{x}} e^{\pi_i^e(y)/\mu} dy}, \ \forall x \ \forall i}$$

Differentiating (both sides of) the logit choice rule (**??**) w.r.t. the decision x establishes that the equilibrium densities satisfy:

$$\pi_i^{e\prime}(x) f_i(x) - \mu f_i'(x) = 0, \ \forall i \tag{0.3}$$

the so-called *logit differential equation* in the equilibrium choice density.

- Note: (**??**) dictates that
  (i) if $\mu \to 0$ then $\pi_i^{e\prime}(x) f_i(x) = 0$; i.e., the condition for an interior NE: either FOC for payoff maximization is satisfied at x else x is chosen with zero probability, $f_i(x) = 0$.
  (ii) if $\mu \to \infty$ then noise effect dominates and $f_i' = 0$; i.e., the equilibrium density is uniform.

Conceptional issues - Rank-based payoffs

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- Rank-based payoffs - payoff depends on whether the decision of the player is 'higher' or 'lower' than the decision(s) of the other player(s)
  Prominent example: auctions
- Consider two-person game: let $\alpha_H(x)$ and $\alpha_L(x)$ be payoff components associated with player i's *own* decision when the decision is higher or lower than the other player's decision.
- Similarly, let $\beta_H(y)$ and $\beta_L(y)$ be payoff components associated with the *other* player's decision when the player i's own decision is higher or lower ranked.

Conceptional issues - Rank-based payoffs

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- Then the payoff function can be written as

$$\pi_i^e(x) = \int_{\underline{x}}^{x} [\alpha_H(x)+\beta_H(y)]f_j(y)dy + \int_{x}^{\bar{x}} [\alpha_L(x)+\beta_L(y)]f_j(y)dy \quad (0.4)$$

- It can be shown that if the component payoff functions are (i) additively separable and (ii) continuous in own, x, and other's, y, decision, then the derivative $\pi_i^{e\prime}(x)$ depends only on the player's own decision x, the other player's distribution and density functions evaluated at x, $F_j(x)$ and $f_j(x)$, and possibly some exogenous parameter ρ.
- Hence, the expected payoff derivative can be written as $\pi_i^{e\prime} = \pi_i^{e\prime}(F_j(x), f_j(x), x, \rho)$, and satisfies the so-called *local payoff property*.

Conceptional issues - Rank-based payoffs

- Extension to n-player case - if one's payoff depends on whether one has the highest (or lowest) decision
  - If having highest is critical (as in an auction), then H and L in (??) represents the case where one's decision is highest or not

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  - Density $f_j(x)$ used in the integrals is replaced by the density of the maximum of the $n-1$ other decisions
  - Example: Second price auction with values $v \rightarrow \alpha_H(x) = v$, $\beta_H(y) = -p_y$, and $\alpha_L(x) = \beta_L(y) = 0$.
- It can be shown that for this case the expected payoff derivative also satisfies the *local payoff property*, i.e., $\pi_i^{e\prime} = \pi_i^{e\prime}(F_{-i}(x), f_{-i}(x), x, \rho)$.

Properties of the Logit equilibrium

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Proposition 1 (Existence) [AGH SEJ02]
There exists a logit equilibrium for all n-player games with a continuum of feasible decisions when players' expected payoffs are bounded and continuous in others' distribution functions. Moreover, the equilibrium distribution is differentiable.

Properties of the Logit equilibrium

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Proposition 2 (Uniqueness) [AGH SEJ02]
Any symmetric logit equilibrium for a game satisfying the local payoff property is unique if the expected payoff derivative, $\pi_i^{e\prime} = \pi_i^{e\prime}(F_{-i}, f_{-i}, x, \rho)$ $\forall i$ *is either*
(a) strictly decreasing in x*, or*
(b) strictly increasing in the common distribution function F*, or*
(c) independent of x *and strictly decreasing in the common density function* f*, or*
(d) a polynomial expression in F*, with no terms involving* f *or* x*.*

Properties of the Logit equilibrium

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Proposition 3 (Comparative Statics for a Symmetric Equilibrium) [Prop. 4, AGH SEJ02]
Suppose that the shift parameter ρ *increases the marginal expected payoffs, i.e.* $\partial\pi_i^{e\prime}(F_{-i}, f_{-i}, x, \rho)/\partial\rho > 0$ $\forall i$*, for a symmetric game satisfying the local payoff property. Then an increase in* ρ *yields stochastically higher logit equilibrium decisions if either*
(a) $\partial\pi_i^{e\prime}/\partial x \leq 0$*, or*
(b) $\partial\pi_i^{e\prime}/\partial F_{-i} \geq 0$*.*

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An experiment!

The Travelers' Dilemma

Basu, AER 1994

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Logit equilibrium - Travelers' dilemma

- The Travelers' dilemma (in continuous strategies)
  - Payoffs: two travelers; each is reimbursed the minimum of the claims in $[\underline{x}, \bar{x}]$ and a fixed penalty $R > 0$ is transfered from the high claimant to the low claimant
  - The penalty gives each traveler an incentive to 'undercut' $\Rightarrow$ both claiming $\underline{\mathbf{x}}$ is the **unique Nash equilibrium** **Independent** of **size** of penalty **R**.

Logit equilibrium - Travelers' dilemma

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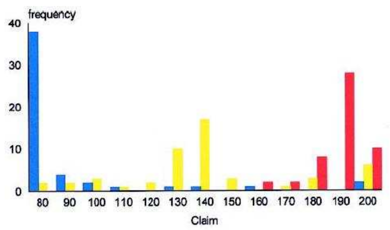
- Travelers' dilemma - the data



blue: R=50, yellow:R=25, red: R=10; frequency of decisions in final 5 of 10 rounds; source: GH PNAS99

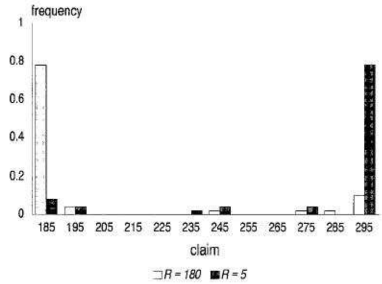


FIGURE 1. CLAIM FREQUENCIES IN A TRAVELER'S DILEMMA FOR R = 180 (LIGHT BARS) AND R = 5 (DARK BARS)

One-shot decisions; source: GH AER01

Logit equilibrium - Travelers' dilemma

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$$\pi_i^e(x) = \int_{\underline{x}}^{x} [\alpha_H(x) + \beta_H(y)] f_j(y) dy \; + \int_{x}^{\bar{x}} [\alpha_L(x) + \beta_L(y)] f_j(y) dy$$

- Travelers' dilemma - Theoretical result

Expected payoff has rank-based property:
Let x be player i's own decision, then
$\alpha_H(x) = -R$, $\alpha_L(x) = x$, $\beta_H(y) = y$, $\beta_L(y) = +R$
Hence, the expected payoff of player i is given by:

$$\pi_i^e(x) = \int_{\underline{x}}^{x} [-R+y] f_j(y) dy + \int_{x}^{\bar{x}} [x+R] f_j(y) dy \; i, j = 1, 2, \; j \neq i$$

And its derivative is:

$$\pi_i^{e\prime}(x) = 1 - F_j(x) - 2R f_j(x) \; i, j = 1, 2, \; j \neq i$$

Logit equilibrium - Travelers' dilemma

$$\boxed{\pi_i^{e\prime}(x) = 1 - F_j(x) - 2Rf_j(x)\ i,j = 1,2,\ j \neq i}$$

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- Travelers' dilemma - Interpretation of expected payoff derivative

  $1 - F_j(x)$ ... probability that the other's claim is higher $\rightarrow$ unilateral increase of claim raises minimum and, hence, the payoff.
  $-2R$ ... payoff reduction when i's claims crosses over j's claim.
  $f_j(x)$ ... probability with which this cross-over occurs.

Logit equilibrium - Travelers' dilemma

$$\boxed{\pi_i^{e\prime}(F_j, f_j, x, -R) = 1 - F_j(x) - 2Rf_j(x)\ i,j = 1,2,\ j \neq i}$$

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- Travelers' dilemma - Property of equilibrium

  Expected payoff derivative has local payoff property.
  Expected payoff derivative increases with parameter $\rho := -R$.
  Expected payoff derivative does not increase with x: $\partial\pi_i^{e\prime}/\partial x = 0$.

  Hence, Proposition **??** implies that a decrease in R yields (stochastically) higher claims.

This is **precisely what we observe**! And, is **not predicted** by the standard **Nash equilibrium**.

Logit equilibrium - Travelers' dilemma

$$\pi_i^{e\prime}(x) f_i(x) - \mu f_i'(x) = 0,\ \forall i$$

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- Travelers' dilemma - Numerical solution

  Substituting the expected payoff derivative into the logit differential equation (see box) gives → second order differential equation in the equilibrium distribution $F(x)$:

$$[1 - F_j(x) - 2Rf_j(x)]f_i(x) - \mu f_i'(x) = 0,\ \forall i, j = 1, 2,\ i \neq j$$

  There exists no analytical solution but it can be solved numerically for given noise μ.

Logit equilibrium - Travelers' dilemma

Slide 40

- Travelers' dilemma - Numerical solutions $\mu = 8.5$ and actual

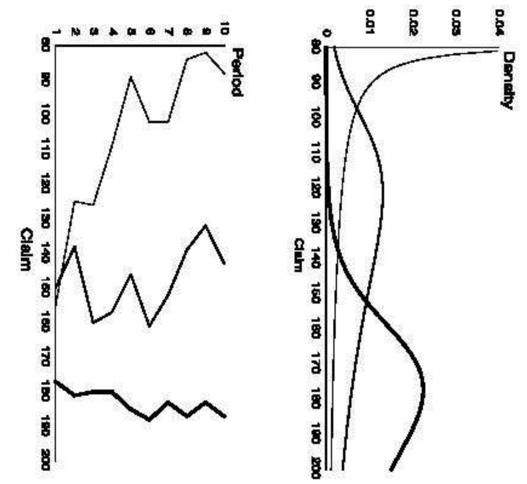

Figure 1. The Traveler's Dilemma.
Key: Predicted Claim Densities (Top) and Actual Claim Averages (Bottom) for $R = 50$ (Left), $R = 20$ (Center), and $R = 10$ (Right)

source: GH PNAS99

Logit equilibrium - Coordination game

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- Minimum-effort coordination game
  - Payoff of a player: minimum effort of all players minus costs c, $0 < c < 1$, of the player's own effort, $x \in [0, \bar{x}]$.
    Player i's expected payoff:

$$\pi_i^e(x) = \int_0^x y f_j(y) dy + \int_x^{\bar{x}} x f_j(y) dy - cx, \; \forall i, j = 1, 2, \; i \neq j$$

    First part: own effort x is above other's effort; second part: own effort x is minimum effort.
    Expected payoff derivative:

$$\pi_i^{e\prime}(F_j, f_j, x, -c) = 1 - F_j(x) - c, \;\; i, j = 1, 2, \;\; j \neq i$$

Logit equilibrium - Coordination game

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$$\boxed{\pi_i^{e\prime}(F_j, f_j, x, -c) = 1 - F_j(x) - c, \;\; i, j = 1, 2, \;\; j \neq i}$$

- Minimum-effort coordination game - Interpretation of expected payoff derivative
  $1 - F_j(x)$ ... probability that other's effort is higher $\rightarrow$ unilateral increase of effort raises minimum effort and, hence, the payoff.
  $-c$ ... cost of raising effort.
- Minimum-effort coordination game - Property of equilibrium
  $\pi_i^{e\prime} > 0 (< 0)$ if $F_j(x) = 0 (= 1) \Rightarrow$ all effort levels are NE.
  Expected payoff derivative increases with parameter $\rho := -c$.
  Expected payoff derivative does not increase with x.

  Hence, Proposition **??** implies that a decrease in effort costs c yields (stochastically) higher effort.

This is **intuitive** but **not predicted** in **Nash equilibrium**.

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Logit equilibrium - Coordination game

- Minimum-effort coordination game - some data

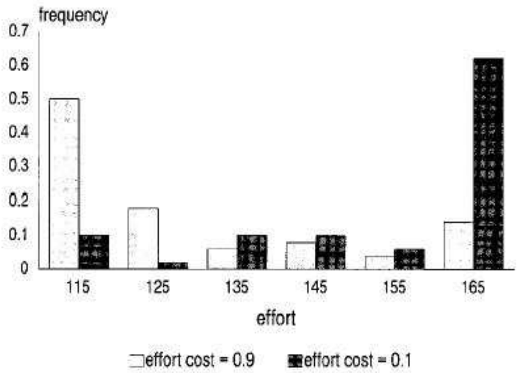

FIGURE 2. EFFORT CHOICE FREQUENCIES FOR A MINIMUM-EFFORT COORDINATION GAME WITH HIGH EFFORT COST (LIGHT BARS) AND LOW EFFORT COST (DARK BARS)

Source: GH AER01

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Logit equilibrium - Public good game

$$\pi_i^e(x) = \int_{\underline{x}}^{x} [\alpha_H(x) + \beta_H(y)] f_j(y) dy \ + \int_{x}^{\bar{x}} [\alpha_L(x) + \beta_L(y)] f_j(y) dy$$

- Public good game
  - Payoff of a player depends on own contribution x_i and on the sum of the other's contribution $\sum_{j \neq i} x_j$:
    Player i's payoff:
    $$\pi_i(x_i) = w_i - x_i + R_I x_i + R_E \sum_{j \neq i} x_j$$
    $R_I < 1$ ... internal return from own contribution,
    R_E ... external return from the sum of other's contribution
    Note: This payoff is a trivial special case of the rank-based payoff; is independent from whether or not one's own contribution is highest or not.

Logit equilibrium - Public good game

$$\pi_i(x_i) = w_i - x_i + R_I x_i + R_E \sum_{j \neq i} x_j$$

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- Public good game - Properties of equilibrium

  Marginal expected payoff from own contribution is constant: $R_I - 1$: Hence,
  Proposition **??** (d) implies **uniqueness**
  The marginal expected payoff is increasing in the exogenous parameter R_I and non-increasing in own contribution x: Hence,
  Proposition **??** (a) implies a stochastic **increase in contributions** with an **increase in** R_I.

This is what we **see in experiments** but is **not predicted** in **Nash equilibrium**.

Combining Inequity aversion and Noise

Goeree & Holt, EER 2000

Slide 46

- Alternating offer bargaining
  - Two-stage alternating offer bargaining game
    Initial pie size $s = 2.40$
    Treatments: different remaining pie sizes after rejection different fixed payments to proposer and responder

Parameters for the 7 bargaining games

Remaining pie size ($R)	0.00	0.40	0.80	1.20	1.60	2.00	2.40
Fixed payment to the initial proposer	2.65	2.25	1.85	1.45	1.05	0.65	0.25
Fixed payment to the initial responder	0.25	0.65	1.05	1.45	1.85	2.25	2.65

Standard SPE: proposer offers R, which is accepted
However, fixed payments **exaggerate earnings inequalities** in equilibrium

Combining Inequity aversion and Noise

Slide 47

- Alternating offer bargaining - Predictions
  - Subgame perfect Nash equilibrium → perfect positive correlation between initial offers and R
  - Final earnings equalization → perfect negative correlation between initial offers and R
  - Note: These relationships might be flattened by noise, but any negative correlation indicates fairness considerations
  - Procedures: 6 sessions, each proposer made 7 proposals, 1 for each treatment in random order, sequentially; then random assignment of proposals to responders, sequentially; accepted or rejected and made counter offer; matched proposer accepted or rejected; all divisions paid out

Combining Inequity aversion and Noise

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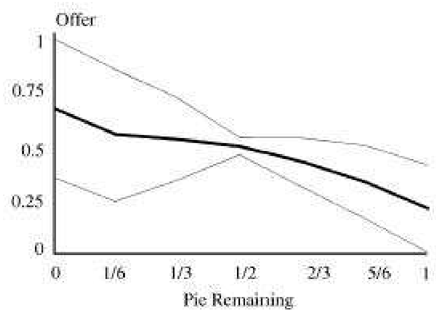
- Alternating offer bargaining - Results



Average 1st stage offers with std.dev.; Source: GH EER00

Clear negative relation between remaining pie size and offers. Standard prediction of positive correlation clearly rejected.
Note 1: Rabin's intention model also predicts positive correlation.
Note 2: The pure Logit model can also not explain the negative correlation.

Combining Inequity aversion and Noise

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- Alternating offer bargaining - FS inequity aversion with noise
  - FS inequity aversion without noise:
    For illustration consider
    $R = 0, FP_R = 2.65, FP_R = 0.25 \Rightarrow \pi_P > \pi_R \; \forall$ offers x:
    $U_P(x) = 2.65 + 2.40 - x - \beta_P[2.40 - 2x]$
    If $\beta > 1/2$: $U'_P(x) > 0 \Rightarrow x^* = 2.40$ (the whole pie)
    If $\beta < 1/2$: $U'_P(x) < 0 \Rightarrow x^*$ depends on α_R but is always less than half of the pie, $2.40/2 = 1.20$.
    Actual data: three offered whole pie, six offered less than half of the pie, but 13 (out of 23) made offers outside predicted range.

Combining Inequity aversion and Noise

Slide 50

- Alternating offer bargaining - FS inequity aversion with noise
  - FS inequity aversion with noise:
    Use logit decision model but replace expected material utility with FS utility:
    Assume all responders have same α_R, β_R, and μ_R
    Assume all proposers have same α_P, β_P, and μ_P
    Estimate these parameters from the data with maximum likelihood technique, to get

Maximum-likelihood estimates

Variable	$\mu_P = \mu_R$	$\alpha_P = \alpha_R$	β_P	β_R
Estimate	0.55	0.84	0.66	0.12
(Standard error)	(0.06)	(0.16)	(0.08)	(0.02)

Source: GH EER00

Combining Inequity aversion and Noise

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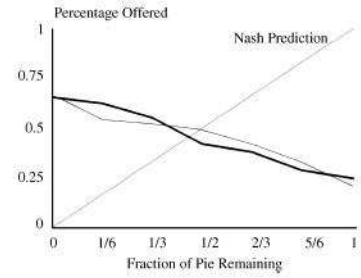
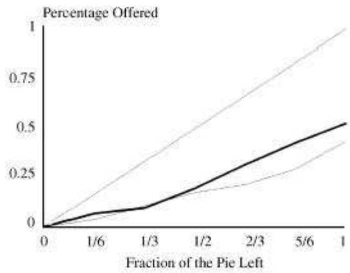
- Alternating offer bargaining - FS inequity aversion with noise



Initial offers;

Source: GH EER00



Counter offers

Source: GH EER00

Thick lines: prediction; thin lines: actual averages

Hence, the model fits the data quite well.