Abstract

Noncooperative games in which each player's payoff function depends on an additively separable function of every player's choice variable may be transformed into an aggregative game, which may be analysed using the concept of 'share functions'. The resulting approach avoids the proliferation of dimensions as the number of players is increased. We show how this approach may be exploited to provide a simple treatment of existence, uniqueness and comparatvie statics in common models that arise in analyses of monopolistic competition, public goods, and rent-seeking contests.

DISGUISED AGGREGATIVE GAMES

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1 Aggregate Games and Share Functions

III: Extensions

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1.1 Three natural questions

1. In view of the advantages of share functions, can they be applied to a larger class of games than those we have defined as 'aggregative'? ['aggregative'≡'players care about an unweighted sum of strategic choices']

2. If so, can we identify the class of games to which the replacement or share function approach can be applied?

3. What interesting applications does this allow us to analyze?

1.2 Three answers

1. Yes

2. Yes

3. Several

1.3 Aggregativeness is not necessary

1.3.1 Why does aggregativeness work?

- The payoff function has the form $\pi_i(q_i, Q)$.
- The FOC [for an interior solution] is

$$\frac{\partial \pi_i(\widehat{q}_i, Q)}{\partial q_i} + \frac{\partial \pi_i(\widehat{q}_i, Q)}{\partial Q}\frac{\partial Q}{\partial q_i} = 0$$

$$\frac{\partial \pi_i(\widehat{q}_i, Q)}{\partial q_i} + \frac{\partial \pi_i(\widehat{q}_i, Q)}{\partial Q} = 0$$

or $\zeta_i(\widehat{q}_i, Q) = 0$

or [assuming an explicit function exists] $\widehat{q}_i = r_i(Q)$.

To summarise: The single aggregate Q contains all the information about the player's economic environment necessary to infer his payoff and his payoff maximising choice. It is a *sufficient statistic* of the game.

1.4 A Simple Example

Consider the following game. Player i's payoff function contains a weighted sum of q_i's:

$$\psi_i(\mathbf{q}) = \left(k - \sum_{j=1}^{n} a_j q_j\right) q_i$$

where $k > 0$ and a_j, $j = 1, ..., n$, are parameters.

Define the aggregate $Q \equiv \sum_{j=1}^{n} a_j q_j$.The payoff function can be written as:

$$\pi_i(q_i, Q) = (k - Q)\, q_i$$

Consider player i's problem. The FOC is

$$\frac{\partial \pi_i(\widehat{q}_i, Q)}{\partial q_i} + \frac{\partial \pi_i(\widehat{q}_i, Q)}{\partial Q} \frac{\partial Q}{\partial q_i} = 0$$

or

$$(k - Q) - \widehat{q}_i a_i = 0$$

$$\therefore \widehat{q}_i = \frac{(k - Q)}{a_i}$$

which is a replacement function!

1.5 A transformation procedure

Here is a formal procedure. Not really needed here, but useful in more general contexts.

Define some new variables: $z_i \equiv a_i q_i$, $i = 1, ..., n$, and $Z \equiv \sum_{j=1}^{n} z_j$. Now substitute, to get

$$\overrightarrow{\psi}_i(\mathbf{z}) = \left(k - \sum_{j=1}^{n} z_j\right) \frac{z_i}{a_i}$$

or

$$\overrightarrow{\pi}_i(z_i, Z) = (k - Z) \frac{z_i}{a_i}$$

Clearly, if every player's payoff function takes this form, the implied game has the aggregative structure and can be analyzed using our suggested procedures.

1.5.1 A necessary requirement

What we need is to be able to determine the player's payoff, and her preferred choice, given the information contained in a single sufficient statistic.

The task is to identify assumptions that are, if possible, necessary and sufficient for this to be the case.

1.6 A Claim

Suppose each player's payoff can be written as

$$\pi_i\left(x_i, t\left(\mathbf{x}\right)\right)$$

where $t\left(\cdot\right)$ is an *additively separable function* of all players' choices, and need not be symmetric. Such a game can be transformed into, and analyzed as, an aggregative game [subject to mild regularity conditions].

1.7 Additive separability is required

Suppose we write $\pi_i(x_i, F(\mathbf{x}))$, but impose no special structure on the aggregator function $F(\mathbf{x})$. Is this enough? No!

Here is an example. 3 players:

$$\psi_i(\mathbf{q}) = \psi_i(q_i, q_1 + q_2 + q_3 + q_1 q_2).$$

or

$$\pi_i(q_i, Q),\ i = 1, ..., n,$$

where $Q \equiv F(\mathbf{q}) = q_1 + q_2 + q_3 + q_1 q_2$.

Player 1's FOC is

$$\frac{\partial \pi_1(\widehat{q}_1, Q)}{\partial q_1} + \frac{\partial \pi_1(\widehat{q}_1, Q)}{\partial Q}\frac{\partial F(\mathbf{q})}{\partial q_1} = 0$$

or

$$\frac{\partial \pi_1(\widehat{q}_1, Q)}{\partial q_1} + \frac{\partial \pi_1(\widehat{q}_1, Q)}{\partial Q}[1 + q_2] = 0$$

Inspection reveals that, in order to determine $\widehat{q}_1$, we need to know more than the value of Q. We need also to know the value of q_2.

To avoid this difficulty, we must assume that $F(\mathbf{q})$ is *additively separable.* Additive separability implies that

$$F(\mathbf{q}) = G\left(\sum_{j=1}^{n} f_j(q_j)\right).$$

The FOC becomes

$$\frac{\partial \pi_1(\widehat{q}_1, Q)}{\partial q_1} + \frac{\partial \pi_1(\widehat{q}_1, Q)}{\partial Q} G'(\cdot) f_1'(\widehat{q}_1) = 0.$$

Recall that $Q = G\left(\sum_{j=1}^{n} f_j(q_j)\right)$. Then the FOC implicitly determines the value $\widehat{q}_1$ in terms of the aggregate $\sum_{j=1}^{n} f_j(q_j)$. No other information is needed to determine $\widehat{q}_1$.

[A formal proof that additive separability is necessary and sufficient is available, but is a bit formal and technical]

1.8

1.9 Three Applications

1. Rent-seeking with nonlinear technology [Tullock, 1980, Cornes and Hartley 2000]

2. Oligopoly with differentiated products[Spence 1976, Dixit and Stiglitz 1977, Blanchard and Kiyotaki 1987]

3. Generalized pure public goods [Cornes 1993, Cornes and Hartley 2000]

1.10 Rent-seeking with nonlinear technology

Recall the basic rent-seeking model, where probability that player i wins is

$$p_i = \frac{x_i}{\sum_j x_j}$$

Tullock (1980) suggests, and various researchers adopt, a generalization:

$$p_i = \frac{x_i^\gamma}{\sum_j x_j^\gamma}$$

where returns to scale are increasing or decreasing according to whether $\gamma >$ or < 1.

Player i's payoff function is

$$\frac{x_i^\gamma}{\sum_j x_j^\gamma}(I_i + R - x_i) + \frac{\sum_{j \neq i} x_j^\gamma}{\sum_j x_j^\gamma}(I_i - x_i)$$

This is not aggregative as it stands.

1.10.1 A useful transformation

Introduce a new variable. Let $z_i = x_i^\gamma$ $\left[\Longrightarrow x_i = z_i^{1/\gamma}\right]$ and substitute. Player i's payoff function becomes

$$\frac{z_i}{\sum_j z_j}\left(I_i + R - z_i^{1/\gamma}\right) + \frac{\sum_{j\neq i} z_j}{\sum_j z_j}\left(I_i - z_i^{1/\gamma}\right)$$

This is now clearly an aggregative game.It can be solved for $Z = \sum_j z_j$ and $z_1, z_2, ..., z_n$. Since there is a one-to-one relationship between z_i and x_i, for all i, it can be solved for the x's.

1.11 A Model of Imperfect Competition

Total demand represented by the quasilinear utility function

$$u(z, \mathbf{q}) = z + h\left(\sum_k q_k^\alpha\right).$$

z is a numeraire good, q_j is output of commodity $j, j = 1, ..., n$.

$h(\cdot)$ everywhere strictly increasing and differentiable.

Commodity j is produced by a single firm, firm j

Total cost function of firm j is $c_j(q_j)$. The inverse demand function for commodity j is

$$p_j = \frac{\partial u(\cdot)}{\partial q_j} = \alpha h'(\cdot)\, q_j^{\alpha-1}$$

In order to transform this into an aggregative game, we introduce the variables x_k and X, where $x_k = f_k(q_k)$ and $X = \sum_j x_j$.

Firm j's profit is

$$\begin{aligned} p_j q_j - c_j q_j &= \alpha h'(\cdot)\, q_j^{\alpha-1} q_j - c_j q_j \\ &= \alpha h'(X)\, x_j - c_j x_j^\alpha \\ &= \psi(X)\, x_j - \chi_j(x_j) \end{aligned}$$

To summarize: the model has been transformed into an undifferentiated product model where X is the total 'utility output' and $\psi(X)$ is its inverse demand function.

If the utility function takes the CES form, then the imperfect competition model may be transformed into an aggregative game which is isomorphic to a model of oligopoly with an undifferentiated product.

1.12 A Generalized Pure Public Good

Recall the standard pure public good model:

player i's preferences: $u_i = u_i(y_i, Q)$,

The 'aggregator function': $Q = \sum_{j=1}^{n} q_j$

Player i's budget constraint :$y_i + q_i \leq m_i$

From a production theory perspective, this aggregator function is very special. The inputs, $q_1, ..., q_n$ are perfect substitutes - a 'Ricardian' technology.

More generally, imagine a general [but separable!] production function:

$$Q = F\left[\sum_{j=1}^{n} f_j(q_j)\right]$$

Player i's problem is

$$\underset{q_i}{\text{Maximize}} \left\{u_i\left[m_i - q_i, F\left(\sum_{j=1}^{n} f_j(q_j)\right)\right]\right\}$$

1.12.1 Another useful transformation

Put $z_i = f_i(q_i)$ $\left[\Longrightarrow q_i = f_i^{-1}(z_i) = g_i(z_i)\right]$ and $Z = \sum_j z_j$.
Player i's problem becomes

$$\underset{z_i}{\text{Maximize}} \ \{u_i [m_i - g_i(z_i), F(Z)]\}$$

or

$$\underset{z_i}{\text{Maximize}} \ \pi_i [z_i, Z]$$

1.13 Another extension - K-reducibility

n players

Player i's preferences: $u_i(y_i, h_i)$

y_i: consumption (a good)

h_i: labour input, measured in hours (a bad)

Player i has an inherent ability or productivity: i's effective labour input, $\ell_i = e_i h_i$, where $e_i > 0$ is an exogenous ability parameter.

$Y = F(L)$: Y total output, $L = \sum_{j=1}^{n} \ell_j$ total input

$F(0) = 0$, $F'(\cdot) > 0$, $F''(\cdot) < 0$.

The output sharing rule:

$$y_i = \frac{h_i}{\sum_{j=1}^{n} h_j} Y = \frac{h_i}{\sum_{j=1}^{n} h_j} F(L)$$

Look at player i's payoff function:

$$u_i(y_i, h_i) = u_i\left(\frac{h_i}{H} F(L), h_i\right) = \pi_i(h_i, H, L)$$

We call this a '2-reducible game'. The player's economic environment can be summarised using two aggregates, L and H.

Nash equilibrium is characterized by two conditions. Let $\widehat{h}_i = r_i(H, L)$ be player i's replacement function. The conditions characterizing Nash equilibrium are

1. $\sum_j \widehat{h}_j = \sum_j r_j(H, L) = H$
2. $\sum_j \widehat{\ell}_j = \sum_j e_j \widehat{h}_j = \sum_j e_j r_j(H, L) = L$

1.13.1 Further examples of K-reducible games

- Public good model with K public goods
- Oligopoly model with K markets - 'island economies'.
- Games in which players care about a finite list of moments of a distribution - for example, (μ, σ).