1 Aggregative Games and Share Functions

II: Applications

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1.0.1

1.0.2 Model 1: Private Provision of Pure Public Goods

The Model There are n players.

Player i's utility function: $u_i(y_i, Q) = u_i(y_i, q_i + Q_{-i})$
$y_i \geq 0$: quantity consumed of the private good
$q_i \geq 0$:player i's contribution to the public good
$u_i(\cdot)$: strictly increasing,strictly quasiconcave, and everywhere differentiable.
Player i's budget constraint: $y_i + c_i q_i \leq m_i$
c_i: the unit cost of the public good.
m_i: player i's exogenous income
Refs: Cornes and Sandler (1985, 1996)
Bergstrom, Blume and Varian (1986)

Player i's optimum Our assumptions imply a unique 'best response' $\widehat{q}_i$ to any given (Q_{-i}, c_i, m_i). This uniquely satisfies the FOC's associated with

$$\underset{y_i, q_i}{\text{Maximize}} u_i \left(y_i, q_i + Q_{-i}\right)$$

subject to $y_i + c_i q_i \leq m_i, y_i \geq 0, q_i \geq 0$

Figure PG1 shows preferences, budget set, and a unique optimum at a point of tangency E. [An optimum could be at either corner, D or F].

Player i's Best Response Function As Q_{-i} varies exogenously, the behaviour of $\widehat{q}_i$ is described by the 'best response function', $\widehat{q}_i(Q_{-i}; c_i, m_i)$.

Two more assumptions:

- y_i and Q are both normal goods
- Given c_i, there is a $Q^{\#}_{-i}$ such that, for all $Q_{-i} \geq Q^{\#}_{-i}$, $\widehat{q}_i = m_i/c_i$.

Properties of the best response function:

- **Continuity**: $\widehat{q}_i(Q_{-i})$ is continuous.
- **Monotonicity**: $\widehat{q}_i(Q_{-i})$ is everywhere nonincreasing.
- **Slope property**: not only is $\widehat{q}_i(Q_{-i})$ monotonic, but at no point does the absolute value of its slope exceed one.

Figure PG2 shows how $\widehat{q}_i$ responds as Q_{-i} varies. Figure PG3 shows the possible shapes of the implied graph of $\widehat{q}_i(Q_{-i})$ in the standard best response diagram.

Player i's Replacement Function Suppose we observe an allocation and we know that, at that allocation, player i is choosing her most preferred bundle. Given an observation Q', what can we infer about $\widehat{q}_i$?

We know that i's chosen bundle is on the line $Q = Q'$. There are three possibilities:

1. $\forall y_i$, $0 \leq y_i \leq m_i$, $\frac{\partial u_i\left(y_i, Q'\right)/\partial Q}{\partial u_i(y_i, Q')/\partial y_i} < c_i$. i's preferred bundle is (m_i, Q') and $\widehat{q}_i = 0$.

2. $\exists \widehat{y}_i$, $0 \leq \widehat{y}_i \leq m_i$, $\frac{\partial u_i\left(\widehat{y}_i, Q'\right)/\partial Q}{\partial u_i(\widehat{y}_i, Q')/\partial y_i} = c_i$. i's preferred bundle is $(\widehat{y}_i, Q')$ and $\widehat{q}_i = (m_i - \widehat{y}_i)/c_i$.

3. $\forall y_i$, $0 \leq y_i \leq m_i$, $\frac{\partial u_i\left(y_i, Q'\right)/\partial Q}{\partial u_i(y_i, Q')/\partial y_i} > c_i$. i's preferred bundle is $(0, Q')$ and $\widehat{q}_i = m_i/c_i$.

In short, there is at most a single value of $\widehat{q}_i$ that is consistent with the given total , Q'. The function that describes how $\widehat{q}_i$ varies in response to changes in Q is player i's **replacement function**, $r_i(Q)$.

Properties of $r_i(Q)$:

- **Domain**:The domain of $r_i(Q)$ is $\left[\underline{Q}, \infty\right)$, where $r_i\left(\underline{Q}\right) = \underline{Q}$.
- **Range**: The range of $r_i(Q)$ is $\left[0, \underline{Q}\right]$
- **Continuity**: $r_i(Q)$ is continuous
- **Monotonicity**: $r_i(Q)$ is everywhere nonincreasing

The three panels of Figure PG4 show the possible shapes of the graph of $r_i(Q)$.

The Aggregate Replacement Function The aggregate replacement function, $r_\Sigma(Q)$, satisfies the following properties:

- **Domain**: $r_\Sigma(Q)$ is defined for all $Q \geq \max_i \left\{\underline{Q}_1, \underline{Q}_1, ..., \underline{Q}_{n1}\right\}$
- **Range**:
  - $r_\Sigma\left(\max_i \left\{\underline{Q}_1, \underline{Q}_1, ..., \underline{Q}_{n1}\right\}\right) \geq 1$
  - There exists a finite value of Q such that $r_\Sigma(Q) = 0$
- **Continuity**: $r_\Sigma(Q)$ is continuous
- **Monotonicity**: $r_\Sigma(Q)$ is everywhere nonincreasing.

$r_\Sigma(Q)$ **and Nash Equilibrium** At a Nash equilibrium, $\sum_j \widehat{q}_j = Q^N$ and $\widehat{q}_i = r_i(Q)$ for all i.

$$\therefore r_\Sigma\left(Q^N\right) \equiv r_1\left(Q^N\right) + r_2\left(Q^N\right) + ... + r_n\left(Q^N\right) = Q^N.$$

Figure PG5 graphs the $r_i(Q)$'s and $r_\Sigma\left(Q^N\right)$ in a 4-player economy.

Q^N is a Nash eq. if the graph of $r_\Sigma\left(Q^N\right)$ intersects the unit slope ray through O at $Q = Q^N$.

Properties 1-4 $\Longrightarrow$ a Nash equilibrium, at which $r_\Sigma(Q) = 1$, exists.

Property 5 $\Longrightarrow$ only one equilibrium exists. Thus :

Existence and Uniqueness The pure public good model possesses a unique Nash equilibrium

Comparative Statics For each player, define the dropout value $\overline{Q}_i$: $\overline{Q}_i \equiv \min(Q \mid r_i(Q) = 0)$. Index individuals according their dropout levels: a type i player drops out at $\overline{Q}_i$, a type j at $\overline{Q}_j$, and $i > j$ if $\overline{Q}_i > \overline{Q}_j$. With this indexing convention, we can state the following:

- At a given equilibrium, if type j players are contributing a positive quantity, so too are all type j players for all $k > j$.
- At a given equilibrium, if type j players are contributing zero, so too are all type j players for all $k < j$.
- Suppose there are many types of player: $t = 1, 2, ..., T$. Then there is a finite number n_T such that, if there are at least n_T players of type T, no other types will make a positive contribution to the public good.

Adding players to the game

- Let an n-player public good game have an equilibrium at $Q^N(n)$. Now allow an extra player - player n+1 - to enter the game, and denote the equilibrium associated with the new game by $Q^N(n+1)$.

Then **either** (i) $r_{n+1}\left[Q^N(n)\right] = 0$, in which case $Q^N(n+1) = Q^N(n)$,
or (ii) $r_{n+1}\left[Q^N(n)\right] > 0$, in which case $Q^N(n+1) > Q^N(n)$.

'Normative' comparative statics See Figure PG6. This shows player i's preference map in (Q, q_i) space. An increase in Q by itself benefits the player, as does a decrease in q_i.The arrows in the figure indicate the direction of more preferred bundles.

Neutrality and Nonneutrality The neutrality proposition: Assume each player faces the same relative prices. *A redistribution of initial incomes between positive contributors to a pure public good that does not change the set of positive contributors has no effect on the resulting equilibrium.*

To verify this proposition consider Figure PG7

Suppose we observe an equilibrium Q'. If i is optimizing, we can infer $q_i = r_i(Q')$ .Now let m_i increase by Δm_i. If Q' is still consistent with payoff maximization by i, the inkind component of income must have fallen by $\Delta m_i / c_i$.

Therefore if i and j are both positive contributors, and $c_i = c_j$, the height of the aggregate replacement function at $Q = Q'$ is unchanged. Q' remains the equilibrium.

1.0.3

1.0.4 Model 2: Sharing Games

The Model n players, each with access to an open access resource - a "fishing ground".

Player i's preferences are represented by $u_i(y_i, \ell_i)$

y_i: i's consumption of a private good

ℓ_i: variable input applied by i to the resource.

$u_i(\cdot)$: quasi-concave, locally nonsatiable, nondecreasing in y_i,nonincreasing in ℓ_i, and everywhere continuous and continuously differentiable.

y_i, ℓ_i are normal

$Y = F(L)$, where $Y = \sum_j y_j$ and $L = \sum_j \ell_j$

$F(L)$ is increasing, strictly concave, continuous and continuously differentiable for $L > 0$, and $F(0) = 0$.

The output sharing rule assigns to player i the quantity $y_i = (\ell_i/L)\,Y$

Player i's Optimum

$$\max_{\ell_i} u_i \left[\left(\frac{\ell_i}{\ell_i + L_{-i}} \right) F \left(\ell_i + L_{-i} \right), \ell_i \right]$$

In this model, the implied best response and replacement functions fail to be monotonic. It is convenient to work with the share, $\widehat{\sigma}_i \equiv \widehat{\ell_i}/L$, as our dependent variable. Here's a sketch of the argument:

FOC for an interior soln:

$$\frac{\partial u_i \left[\frac{\ell_i F(L)}{L}, \ell_i \right] / \partial \ell_i}{\partial u_i \left[\frac{\ell_i F(L)}{L}, \ell_i \right] / \partial y_i} = \frac{\ell_i}{L} F' \left(L \right) + \left(1 - \frac{\ell_i}{L} \right) \frac{F \left(L \right)}{L}$$

or

$$\begin{aligned} \xi_i \left(\sigma_i, L \right) &= \tau_i \left(\sigma_i, L \right) \\ \text{or } MRS_i &= MRT_i \text{ !} \end{aligned}$$

Properties of the MRS and MRT functions It may be confirmed that $\partial \xi_i(\cdot)/\partial \sigma_i > 0, \partial \xi_i(\cdot)/\partial L > 0$

and $\partial \tau_i(\cdot)/\partial \sigma_i < 0, \partial \tau_i(\cdot)/\partial L < 0$.

These observations imply

- For any given L, there is at most a single payoff maximising $\widehat{\sigma}_i$ [that is, a share function exists]
- $\widehat{\sigma}_i = s_i(L)$ is a decreasing function of L. [strictly decreasing wherever $\widehat{\sigma}_i > 0$]

See Figure SG1. Recall the characterization of a Nash equilibrium in terms of share functions: L^N is a Nash equilibrium iff $s_\Sigma\left(L^N\right) \equiv \sum_j s_j\left(L^N\right) = 1$.

See Figure SG2. Again we can infer

Existence and Uniqueness The sharing game possesses a unique Nash equilibrium.

1.0.5

Comparative static results

- If $L^2 > L^1$ and $s_i\left(L^1\right)$ exists, (i) $L^1\left[1 - s_i\left(L^1\right)\right] < L^2\left[1 - s_i\left(L^2\right)\right]$ and (ii) $\omega_i\left(s_i\left(L^1\right), L^1\right) > \omega_i\left(s_i\left(L^2\right), L^2\right)$, where $\omega_i(\cdot)$ is i's payoff in terms of $s_i(L)$ and L.

- Form the open access game G^2 by taking the game G^1 and adding extra players, so that $I^1 \subset I^2$. Then
  - (i) $L^{1N} \leq L^{2N}$, [“<” if some player not in I^1 makes a positive contribution in G^2.
  - (ii) $L^{1N}\left[1 - s_i\left(L^{1N}\right)\right] < L^{2N}\left[1 - s_i\left(L^{2N}\right)\right]$ for all $i \in I^1$[“<” if $L^{1N} < L^{2N}$]
  - (iii) Nash eq payoffs to all players in I^1 are no larger in G^2 than in G^1. If $L^{2N} > L^{1N}$, payoffs are strictly less for those who make strictly positive contributions in G^1.

- Initially, suppose an eq L^N. Introduce a uniform quota on the level of individual inputs of the form $\ell_i \leq \ell^{\max}$ for all $i \in I$. Then at the regulated eq, L^{RN},
  - (i) $L^{RN} \leq L^N$
  - (ii) For any player not facing a binding constraint in the regulated equilibrium, $L^{RN}\left[1 - s_i\left(L^{RN}\right)\right] < L^N\left[1 - s_i\left(L^N\right)\right]$
  - (iii) Nash eq payoffs to all players not facing a binding quota at the regulated eq are no less than at the unregulated eq, and are strictly higher if the quota binds for at least one player.
- If player i's inherent productivity increases:
  - (i) $L^{2N} \geq L^{1N}$
  - (ii) Either $L^{2N} = L^{1N}$ and $s_i\left(L^{2N}\right) = s_i\left(L^{1N}\right) = 0$, or $L^{2N} > L^{1N}$ and $s_i\left(L^{2N}\right) > s_i\left(L^{1N}\right) \geq 0$
  - (iii) Player i's payoff will not fall, and will rise if $s_i\left(L^{2N}\right) > 0$
  - (iv) Other players' payoffs will not rise, and will fall if $s_i\left(L^{2N}\right) > 0$.

Some natural extensions

- To other surplus sharing rules - for example, $x_i = \theta_i X$, where $\sum_j \theta_j = 1$, or even to a convex combination of exogenous and proportional sharing rules: $X^E = \lambda X$, $X^P = (1 - \lambda) X$, and $x_i = \theta_i X^E + \frac{\ell_i}{L} X^P$
- To cost sharing rules: applications to banking clearing schemes, sharing airport costs, or other shared facilities.

1.0.6

1.0.7 Model 3: A Rentseeking Contest

The Model n players

Exogenous rent, R

x_i: effort devoted by player i to contesting the rent

I_i: player i's initial wealth

Either $u_i(W_i) = -e^{-\alpha_i W_i}$: $\alpha_i > 0$ player i's risk aversion coefficient

or $u_i(W_i) = W_i$ [Player i is risk neutral]

Probability that player i wins the rent: $p_i = \frac{x_i}{X} = \frac{x_i}{\sum_j x_j}$

Player i's payoff function:

$$\pi_i(x_i, X) = \frac{x_i}{X}\left\{-e^{-\alpha_i[I_i+R-x_i]}\right\} + \left(1 - \frac{x_i}{X}\right)\left\{-e^{-\alpha_i[I_i-x_i]}\right\}$$

Player i's share function Derivation of FOC, followed by rearrangements, yields player i's share function:

$$s_i(X) = \max\left\{\frac{1-\beta(\alpha_i)X}{1-\alpha_i X}, 0\right\}$$

where

$$\beta(\alpha_i) = \frac{\alpha_i}{(1-e^{-\alpha_i R})} \text{ for } \alpha_i > 0 \text{ and } \beta(0) = 1/R.$$

Properties of the share function

- (i) $s_i(X) \longrightarrow 1$ as $X \longrightarrow 0$
- (ii) $s_i(X)$ is continuous
- (iii) For all values of $X < 1/\beta(\alpha_i)$, $s_i(X) > 0$ and strictly decreasing
- (iv) For all values of $X \geq 1/\beta(\alpha_i)$, $s_i(X) = 0$

See Figure RS1. Again we can immediately infer

Existence and Uniqueness The sharing game possesses a unique Nash equilibrium.

Comparative statics

- If each of n players is risk-neutral or risk averse with common $\alpha > 0$, then as $n \longrightarrow \infty$, $X/R \longrightarrow \rho(\alpha)$, where $\rho(\alpha) = \frac{1-e^{-\alpha R}}{\alpha R}$[which implies $\rho(0) = 1$ and $\lim\limits_{\alpha \longrightarrow \infty} \rho(\alpha) = 0$]

- If $\underline{\alpha} = \inf\limits_{i=1,\ldots,\infty} \alpha_i$ and $\alpha_i > \underline{\alpha}$ for all i, $X/R \longrightarrow \rho(\underline{\alpha})$ as $n \longrightarrow \infty$.

- If all players are strictly risk averse and $\inf\limits_{i=1,\ldots,\infty} \alpha_i = 0$, then rent is fully dissipated in the limit.

- Let F be a distribution function on the non-negative real numbers and let $\underline{\alpha} \geq 0$ be the greatest lower bound of its support. Let $\alpha_1, \alpha_2, \ldots$ be independent draws from this distribution. Then

$$\lim_{n \longrightarrow \infty} \frac{X^{(n)}}{R} = \rho(\underline{\alpha}) \text{ with probability 1.}$$