# 1 Aggregative Games and Share Functions

I: Theory

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### 1.0.1

### 1.0.2 Economics and Interdependence

Gorman, W. M. (1976), "Tricks with Utility Functions"
Gorman's two tricks:

1. Duality: "Duality is about the choice of the independent variables in terms of which one defines a theory"

2. Separability: "Separability has to do with the structure of a problem"

Like Gorman, these lectures argue that one's life can be greatly simplified if, at the outset of a modelling exercise, one gives careful thought to the 'natural structure' of a problem, and to the 'natural independent variables'.

### 1.0.3 Static Noncooperative Game Theory

Definition of Nash noncooperative equilibrium [Nash (1951)]:

Let $x_i$ be player $i$'s strategy in an $n$-player game. A Nash noncooperative equilibrium is a set of strategies $(x_1^*, x_2^*, ..., x_n^*)$ such that, for every $i = 1, ..., n$,

$$u_i\left(x_1^*, x_2^*, ..., x_i^*, ..., x_n^*\right) \geq u_i\left(x_1^*, x_2^*, ..., x_i, ..., x_n^*\right)$$

for every feasible strategy

[Gibbons, (1992), p8]

As a definition of a social equilibrium, this is beyond reproach and has proved remarkable fruitful.

As a starting point for formal analysis, this way of describing equilibrium quickly runs into difficulties.

### 1.0.4 Some Difficulties

- Geometric representation runs into dimensional problems
- Arguments that work for small values of n break down for larger values

### 1.0.5 Common Responses

- Restriction to 2-player models
- Assumption that players are identical and Nash equilibrium is symmetric

### 1.0.6 Three claims

1. Many, perhaps most, economic models that use noncooperative game theory have a lot of special structure

2. This structure can be exploited to provide alternative ways of thinking about equilibrium.

3. Some of these alternatives permit a dramatic simplification of the analysis. By simplifying, we can

   - gain insights into the economic intuition of the models
   - extend the scope of models that are analytically tractable
   - address questions that have not been - and, arguably, cannot be - addressed using the standard approach
   - provide a remarkably simple graphical exposition

### 1.0.7 Aggregative Games

Definition: An $n$-player game is aggregative if the payoff of every player can be expressed as a function of two variables: (i) that player's own choice variable, and (ii) the unweighted sum of every player's choice variable:

$$u_i(q_1, q_2, ..., q_{i-1}, q_i, q_{i+1} ..., q_n) = v_i(q_i, Q)$$

where $Q \equiv \sum_{j=1}^{n} q_j$.

**Examples**

- **Cournot oligopoly**: Let the inverse demand function for a good be $P = P(Q)$, where Q is the total output of the good. Each of n producers has total cost function $C_i(q_i)$. Each firm seeks to maximize its profit, given the output levels of the others. Then the payoff, or profit, of firm i is

  $$\pi_i(\cdot) \equiv P(Q) q_i - C_i(q_i)$$

  Clearly, if every firm's profit function is of this form, the game is aggregative.

- **Pure Public Good**: Player $i$ in an $n$-player model cares about the quantity of consumption of a private good, $x_i$, and the total available quantity of a public good. Each player has an exogenous income, $m_i$. Each unit of private good costs $p_x$, the provision of each unit of the public good costs its contributor $p_q$, and each player allocates the whole of her income to either private good consumption or public good contribution, so that $p_x x_i + p_q q_i = m_i$. For given values of income and of prices, player i's payoff may be written as

$$u_i\left(x_i, Q\right) = u_i\left(\frac{m_i - p_q q_i}{p_x}, Q\right) = \upsilon_i\left(q_i, Q\right).$$

  Clearly, this game is aggregative.

- **Impure Public Good**:

$$\begin{aligned} u_i\left(x_i, q_i, Q\right) &= u_i\left(\frac{m_i - p_q q_i}{p_x}, q_i, Q\right) \\ &= \upsilon_i\left(q_i, Q\right) \end{aligned}$$

- **The Problem of the Commons**: Each of $n$ players chooses how much labour, $\ell_i$, to apply to an unpriced resource - say a fishing ground. The total catch, $Q$, depends upon the total amount of labour applied, $L = \sum_{j=1}^{n} \ell_j$: $Q = F(L)$. The proportion of the total catch that is taken by player $i$ is $\ell_i / L$. Each playercares about the number of fish available for his consumption and also the amount of time that he devotes to fishing:

$$\begin{aligned} u_i(x_i, \ell_i) &= u_i\left(\frac{\ell_i}{L} Q, \ell_i\right) \\ &= u_i\left(\frac{\ell_i}{L} F(L), \ell_i\right) = v_i(\ell_i, L). \end{aligned}$$

- **The Cooperative Enterprise**: Suppose, instead, that the n fisherman of the last example choose to share out their total catch equally. If the total catch is $Q$, each receives the quantity $Q/n$ for consumption. Player $i$'s payoff is now

$$\begin{aligned} u_i(x_i, \ell_i) &= u_i\left(\frac{Q}{n}, \ell_i\right) \\ &= u_i\left(\frac{F(L)}{n}, \ell_i\right) = \overrightarrow{v}_i(\ell_i, L). \end{aligned}$$

  Again, the game is aggregative. We have placed an arrow over the $v$ simply to remind you that this is not precisely the same function as the one that appeared under the sharing rule of the previous problem. This, like the 'problem of the commons', is an example of a **surplus sharing model**. There are also

- **Costsharing Models**

- **A Rent-seeking Contest**: There is a given prize of $R$. Each of $n$ players spends resources competing for the rent. If player $i$ expends the level $x_i$, the probability that $i$ wins the rent is given by

$$p_i = \frac{x_i}{\sum_{j=1}^{n} x_j} = \frac{x_i}{X}.$$

Player $i$'s initial wealth is $I_i$, and each wants to maximize the expected value of his wealth. Player $i$'s payoff is therefore

$$\begin{aligned} u_i(x_i, Q) &= p_i(I_i + R - x_i) + (1 - p_i)(I_i - x_i) \\ &= \frac{x_i}{X}(I_i + R - x_i) + \left(1 - \frac{x_i}{X}\right)(I_i - x_i) \\ &= \upsilon_i(x_i, X). \end{aligned}$$

### 1.0.8 Model 1

**A Simple Duopoly Model** Two firms. Firm $i$'s output level is $q_i$ and its unit cost is $c_i$. Note that we allow the firms to have different cost levels.

The inverse demand function is

$$\begin{aligned} P(Q) &= \alpha - \beta Q \ \text{ for } Q \leq \alpha/\beta \\ P(Q) &= 0 \text{ for } Q > \alpha/\beta \end{aligned}$$

where $\alpha$ and $\beta$ are positive parameters.

Profit of firm $i$:

$$\begin{aligned} \psi_i(q_1, q_2) &= [\alpha - \beta(q_1 + q_2)]\, q_i - c_i q_i \\ i, j &= 1, 2 \quad i \neq j \end{aligned}$$

A Nash noncooperative equilibrium is a pair, $(q_1^*, q_2^*)$, such that
$q_1^*$ maximizes firm 1's profits given that firm 2 is producing $q_2^*$, and
$q_2^*$ maximizes firm 2's profits given that firm 1 is producing $q_1^*$.

**Necessary and Sufficient Conditions** First-order necessary conditions [sufficient as well] are, for firm 1,

$$\alpha - 2\beta\widehat{q}_1 - \beta q_2 - c_1 \leq 0, \quad \widehat{q}_1 \geq 0,$$

$$[\alpha - 2\beta\widehat{q}_1 - \beta q_2 - c_1]\,\widehat{q}_1 = 0$$

and, for firm 2,

$$\alpha - 2\beta\widehat{q}_2 - \beta q_1 - c_2 \leq 0, \quad \widehat{q}_2 \geq 0,$$

$$[\alpha - 2\beta\widehat{q}_2 - \beta q_1 - c_2]\,\widehat{q}_2 = 0$$

The alternative ways of analyzing the game involve different ways of rearranging these conditions, both of which must hold at a Nash equilibrium. We look first at the 'best response function' approach.

**Best Response Functions** Rearrange the FOC's so that each firm's output is expressed as an explicit function of the other firm's. This produces the 'best response' functions:

$$\widehat{q}_1 = \widehat{q}_1 \left( q_2 \right) = \max \left\{ \frac{\alpha - c_1 - \beta q_2}{2\beta}, 0 \right\}$$

and

$$\widehat{q}_2 = \widehat{q}_2 \left( q_1 \right) = \max \left\{ \frac{\alpha - c_2 - \beta q_1}{2\beta}, 0 \right\}.$$

**The Replacement Function of a Player**

1. Consider again player 1's FOC. For the moment, ignore nonnegativity constraints. Observe that, at any allocation, $q_2 = Q - q_1$ by definition. Thus player 1's FOC may be written as

$$2\beta\widehat{q}_1 = \alpha - c_1 - \beta\left(Q - \widehat{q}_1\right).$$

This restates the information contained in the best response function, but in the form of an implicit function involving $\widehat{q}_1$ and $Q$. It can be rearranged to express the individual's most preferred choice as an explicit function of $Q$:

$$\widehat{q}_1 = r_1\left(Q\right) = \frac{\alpha - c_1 - \beta Q}{\beta}.$$

**Two nonnegativity issues**

- if $Q > (\alpha - c_1)/\beta$, the FOC as written above does not have a nonnegative solution in $\widehat{q}_1$. Thus, like the best response function, the replacement function is piecewise linear. As $Q$ increases, $\widehat{q}_1$ falls until it reaches zero at $Q = (\alpha - c_1)/\beta$ . Thereafter it remains zero as $Q$ increases further.

- An economically meaning solution must be consistent with the requirement that $\widehat{q}_1 \leq Q$. This restricts the admissible domain of the function, since it requires that $Q \geq (\alpha - c_1)/2\beta$. Taking these restrictions into account, we write the 'replacement function of player 1' as

$$
\begin{aligned}
\widehat{q}_1 &= r_1(Q) \\
&= \max\left\{\frac{\alpha - c_1 - \beta Q}{\beta}, 0\right\}
\end{aligned}
$$

for $Q \geq (\alpha - c_1)/2\beta$.

**Significant properties of $r_1(Q)$**

1. At $Q = (\alpha - c_1)/2\beta$, $r_1(Q) = Q$.
2. At $Q = (\alpha - c_1)/\beta$, $r_1(Q) = 0$.
3. $r_1(Q)$ is everywhere continuous. [In this example it is piecewise linear] .
4. $r_1(Q)$ is everywhere nonincreasing, and at every point where $r_1(Q) > 0$, it is strictly decreasing in $Q$.

These simple observations play a key role in the analysis that follows.

**1.0.9**

### 1.0.10 Model 2

Same as model 1, except that the demand function, instead of being linear, is isoelastic [indeed, unit elastic]:

$$P(Q) = 1/Q$$

so that firm $i$'s payoff function is

$$\begin{aligned} \pi_i(\cdot) &= P(Q)\, q_i - c_i q_i \\ &= \frac{q_i}{Q} - c_i q_i. \end{aligned}$$

Player 1's optimizing problem is

$$\max_{q_1} \left\{ \frac{q_1}{q_1 + q_2} - c_1 q_1 \right\}.$$

A necessary and sufficient condition for $\widehat{q}_1 > 0$ to maximize the payoff is

$$c_1 \widehat{q}_1^2 + 2 c_1 q_2 \widehat{q}_1 + \left(c_1 q_2^2 - q_2\right) = 0.$$

**The Best Response Function of a Player** Restricting attention to the positive solution of this quadratic equation in $q_1$, and taking account of the possibility that the nonnegativity constraint on $q_1$ may be binding, player 1's best response function is

$$\widehat{q}_1 = \max\left\{\sqrt{\frac{q_2}{c_1}} - q_2, 0\right\}.$$

Note that this best response function is not only nonlinear, but non-monotonic. As $q_2$ rises, so too does firm 1's response up to the point where $q_2 = 1/4c_1$. From this point onwards, $\widehat{q}_1$ declines as $q_2$ increases until $q_2 = 1/c_1$. Thereafter, as $q_2$ rises further, $\widehat{q}_1 = 0$.

**The Replacement Function of a Player** The replacement function for player may be obtained by replacing the quantity $q_2$ by $(Q - \widehat{q}_1)$ in the best response function. Doing this and rearranging the resulting expression to provide an explicit function for $\widehat{q}_1$, we obtain

$$\widehat{q}_1 = \max\left\{Q - c_1 Q^2, 0\right\} \text{ for } Q > 0.$$

This is the replacement function of player 1 in the present example. Like the best response function, it is nonlinear and non-monotonic.

If we wished, we could use the players' replacement functions to characterize Nash equilibrium and explore its properties. However, the fact that it is non-monotonic creates slight additional difficulties. For this reason, we prefer to work with a slightly different representation of players' behavior, using what we call the 'share function'.

**The Share Function of a Player** The replacement function differs from the best response function in that it uses the total $Q$ as the independent variable. The share function shares this feature, but works with a different dependent variable. For all strictly positive values of $Q$, we can divide both sides of the replacement function by $Q$. The result is an equation whose dependent variable is the player's share of total output:

$$\frac{\widehat{q}_1}{Q} = \max\left\{1 - c_1 Q, 0\right\} \text{ for } Q > 0.$$

Inspection immediately reveals an attractive property of $s_i(Q)$ in the present example - it is piecewise linear.

**Share Functions and Nash Equilibrium** If the quantity $Q^*$ is a Nash equilibrium, the following must hold:

1. it must be the case that each player's chosen quantity, expressed as a proportion or share of the total Q, is described by that player's share function:
$$\frac{\widehat{q}_1}{Q^*} = s_1(Q^*) = \max\{1 - c_1 Q^*, 0\}$$
and $\frac{\widehat{q}_2}{Q^*} = s_2(Q^*) = \max\{1 - c_2 Q^*, 0\}$.

2. The sum of the players' share values must equal one:

$$\begin{aligned} \widehat{q}_1 + \widehat{q}_2 &= Q^* \\ &\implies \frac{\widehat{q}_1}{Q^*} + \frac{\widehat{q}_2}{Q^*} = 1 \\ &\implies s_1(Q^*) + s_2(Q^*) = 1. \end{aligned}$$

**Properties of equilibrium**

- Existence of equilibrium: Clearly, for sufficiently small values of $Q$, the sum of players' share values exceeds one. Further, for sufficiently large values of $Q$, the sum of players' share values falls short of one. Since the aggregate share function is piecewise linear, it is continuous. Therefore the existence of a value, $Q^*$, at which the sum of players' shares precisely equals unity is ensured.

- Uniqueness of equilibrium: The aggregate share function is everywhere strictly decreasing wherever its value is strictly positive. Therefore there can only be one value $Q^*$ at which $s_1(Q^*) + s_2(Q^*) = 1$.

- Comparative statics: easy to do by considering how shocks shift individual, and therefore the aggregate, share functions.