Lecture 1–A Primitive Public Economy

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Chapter 1

A Primitive Public Economy

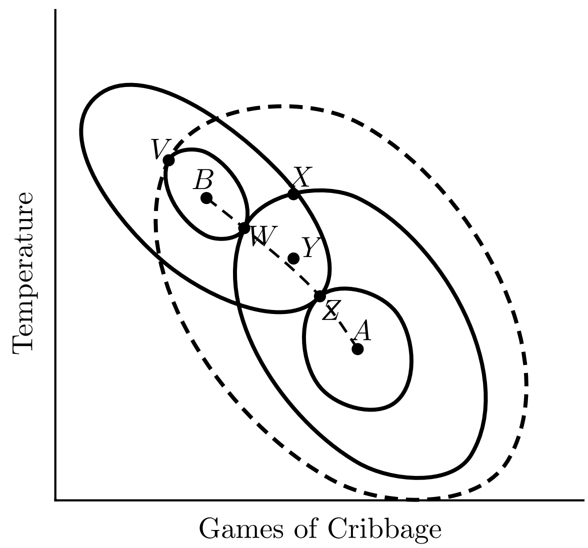
Anne and Bruce are roommates. They are interested in only two things; the temperature of their room and playing cribbage together. Each of them has a different favorite combination of room temperature and games of cribbage per week. Anne's preferred temperature may depend on the number of games of cribbage that she is allowed to play per week and her preferred number of games of cribbage may depend on the room temperature. Given the number of games of cribbage, the further the temperature deviates from her favorite level, the less happy she is. Similarly, given the temperature, Anne is less happy the more the number of games of cribbage differs from her preferred number. Bruce's preferences have the same qualitative character as Anne's, but his favorite combination is different from hers.

The landlord pays for the cost of heating their room and the cost of a deck of cards is negligible. Since there are no scarce resources in the usual sense, you might think that there is not much here for economists to study. Indeed if Anne lived alone and her only choices involved temperature and *solitaire*, the economic analysis would be pretty trivial. She would pick her bliss point and that's that.

The tale of Anne and Bruce is economically more interesting because althought they may disagree about the best temperature and the best amount of cribbage-playing, each must live with the same room temperature and (since they are allowed no other game-partners) each must play the same number of games of cribbage as the other. Somehow they will have to settle on an outcome in the presence of conflicting interests. This situation turns out to be a useful prototype for a wide variety of problems in public economics.

We begin our study with an analysis of efficient conduct of the Anne–

Figure 1.1: **Indifference Curves for Anne and Bruce**



Bruce household. A diagram will help us to understand how things are with Anne and Bruce. In Figure 1.1, the points A and B represent Anne's and Bruce's favorite combinations of cribbage and temperature. These points are known as Anne's and Bruce's bliss points, respectively. The closed curves encircling A are indifference curves for Anne. She regards all points on such a curve as equally good, while she prefers points on the inside of her indifference curves to points on the outside. In similar fashion, the closed curves encircling B are Bruce's indifference curves.

We shall speak of each combination of a room temperature and a number of games of cribbage as a *situation*. If everybody likes situation α as well as situation β and someone likes α better, we say that α is *Pareto superior* to β. A situation is said to be *Pareto optimal* if there are no possible situations that are *Pareto* superior to it. Thus if a situation is not Pareto optimal, it should be possible to obtain unanimous consent for a beneficial change. If the existing situation is Pareto optimal, then there is pure conflict of interest in the sense that any benefit to one person can come only at the cost of harming another.

Our task is now to find the set of Pareto optimal situations, *chez Anne and Bruce*. Consider a point like X in Figure 1.1. This point is not Pareto optimal. Since each person prefers his inner indifference curves to his outer ones, it should be clear that the situation Y is preferred by both Anne and Bruce to X. Anne and Bruce each have exactly one indifference curve passing through any point on the graph. At any point that is not on boundary of the diagram, Anne's and Bruce's indifference curves through this point either cross each other or are tangent. If they cross at a point, then, by just the sort of reasoning used for the point X, we see that this point can not be Pareto optimal. Therefore Pareto optimal points must either be points at which Anne's indifference curves are tangent to Bruce's or they must be on the boundary of the diagram.

In Figure 1.1, all of the Pareto optimal points are points of tangency between Anne's and Bruce's indifference curves. Points Z and W are examples of Pareto optima. In fact there are many more Pareto optima which could be found by drawing more indifference curves and finding their tangencies. The set of such Pareto optima is depicted by the line BA in Figure 1.1. Although every interior Pareto optimum must be a point of tangency, not every interior point of tangency is a Pareto optimum. To see this, take a look at the point V on the diagram. This is a point of tangency between one of Anne's indifference curves and one of Bruce's. But the situation V is not Pareto optimal. For example, both Anne and Bruce prefer B to V. In our later discussion we will explain mathematical techniques that enable you to distinguish the "good" tangencies, like Z and W, from the "bad" ones, like V.

Let us define a person's *marginal rate of substitution* between temperature and cribbage in a given situation to be the slope of his indifference curve as it passes through that situation. From our discussion above, it should be clear that at an interior Pareto optimum, Anne's marginal rate of substitution between temperature and cribbage must be the same as Bruce's. If we compare a Pareto optimal tangency like the point Z in Figure 1.1 with a non–optimal tangency like the point V, we notice a second necessary condition for an interior Pareto optimum. At Z, Anne wants more cribbage and a lower temperature while Bruce wants less cribbage and a higher temperature. At V, although their marginal rates of substitution are the same, both want more cribbage and a lower temperature. Thus a more complete necessary condition for a Pareto optimum is that their marginal rates of substitution be equal and their preferred directions of change be opposite.

The Utility Possibility Frontier and the Contract Curve

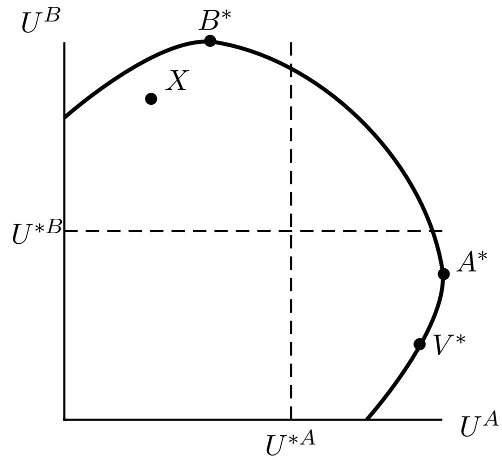
With the aid of Anne and Bruce we can introduce some further notions that are important building blocks in the theory of public decisions.

The Utility Possibility Set and the Utility Possibility Frontier

Suppose that Anne and Bruce have utility functions $U^A(C, T)$ and $U^B(C, T)$, representing their preferences over games of cribbage and temperature. We can graph the possible distributions of utility between them. On the horizontal axis of Figure 1.2, we measure Anne's utility and on the vertical axis we measure Bruce's utility. Each possible combination of temperature and number of games of cribbage determines a possible distribution of utility between Anne and Bruce. The *utility possibility set* is defined to be the set of all possible distributions of utility between Anne and Bruce. The *utility possibility frontier* is the "northeast" (upper right) boundary of this set. A point like X in Figure 1.1 that is not Pareto optimal would correspond to a point like X^* in Figure 1.2 that is not on the utility possibility frontier. The point A^* in Figure 1.2 represents the utilities for Anne and Bruce achieved from Anne's favorite position (A in Figure 1). Similarly, B^* represents the utilities achieved from Bruce's L favorite position. The curved line A^*B^* in Figure 1.2 is the "utility possibility frontier".

It is interesting to interpret the meaning of the entire boundary of the utility possibility set. Notice that it is impossible to make Anne any better off than she is at her bliss point. Therefore, the rightmost point that the utility possibility frontier attains is the point A^*. If Bruce is to be made better off than he is at Anne's bliss point, then Anne will have to be made worse off. Below A^*, are points where Bruce is worse off than he would be at Anne's bliss point. Since Anne and Bruce share the same environment, if Bruce is to be worse off than he is at Anne's bliss point, Anne must be worse off as well. Thus, below the point A^*, the boundary of the utility possibility frontier must slope upward. Recall the point V, on Figure 1.1, where although Anne's indifference point is tangent to Bruce's, situation V is not Pareto optimal. But V does correspond to a point on the southeast (lower-right) boundary of the utility possibility set. In particular, although it is possible to make Anne and Bruce simultaneously better off by moving away from V, we see that Bruce is on the highest indifference curve he can attain if we insist that Anne is to be left on the *same* indifference curve as V. Therefore the situation depicted by V would correspond to a point

Figure 1.2: A Utility Possibility Frontier



on the upward-sloping boundary of the utility possibility set like V^*. The situation V might be of interest to someone (perhaps Anne's inlaws?) who liked Bruce but hated Anne.[1]

By the same kind of reasoning, we argue that to the left of the point B^*, the boundary of the utility possibility frontier slopes upward from right to left. This means that making Anne worse off than she is at Bruce's bliss point will be costly to Bruce.[2]

In general, the utility possibility set need not be a convex set. In fact it could be of almost any shape. But, by construction, the utility possibility frontier is the part of the boundary of the utility possibility set that slopes downward and to the right. One question that may have occurred to you is the following. We know that if someone's preferences can be represented by one utility function, then these same preferences can also be represented by any monotonic transformation of that function.

Sometimes this idea is expressed by saying that representation of preferences by utility functions is unique only up to monotonic transformations.

[1]In other contexts, such points may be of interest, because they represent the "cost" to Anne of various "threats" that she might make in the course of bargaining.

[2]Whether the utility possibility set is bounded from below in the lower left quadrant depends on whether the utility functions are bounded from below or whether it would be possible to make Anne and/or Bruce arbitrarily "miserable" by, say, making the temperature and the number of games sufficiently high.

But the shape of the utility possibility frontier will in general depend on *which* monotonic transformation you use. This is true. You have to first specify the utility representation that you intend to use and then draw the utility possibility frontier. [3]

Reservation Utilities and the Contract Curve

One thing that we haven't discussed so far is the possibility that either Anne or Bruce might have some options other than living in the Anne-Bruce household. Either of them might choose to live alone, or perhaps find an alternative partner. Let us denote the best utility level that Anne could achieve from an alternative living arrangement by U^{*A} and the best alternative level that Bruce could achieve by U^{*B}. These are known as the *reservation utilities* for Anne and Bruce. Arrangements in the household must be such that Anne gets at least her reservation utility or she will move out. Similarly for Bruce. The part of the utility possibility frontier that lies above and to the right of the two dotted lines in Figure 1.2 is known as the *contract curve* between Anne and Bruce.

Notice that if Anne and Bruce had high enough reservation utilities, there might be no points on the contract curve for them. In this case, there would be no way that they could live together and both be as well off as if they would be if they exercised their outside options.

Some Lagrangean Housekeeping

In order to generalize our theory to more people and more commodities, we need more powerful tools. Among the tools that we will find useful are the method of Lagrange multipliers and its extension to problems with inequality constraints, the Kuhn-Tucker theory. As it happens, we can conduct an entirely satisfactory analysis of Anne's and Bruce's little household using only graphical methods. This is no accident. The example was very carefully chosen to lend itself to graphing. As soon as we want to study even slightly more complex environments, we find that graphical methods are not able to handle all of the relevant variables in neat ways. To enter this larger domain, we need to be equipped with Lagrangian and Kuhn-Tucker methods. In later

[3] When we turn to the discussion of gambles and uncertainty, we will find that the most useful representations of utility are limited to a family that is 'unique up to linear transformations' and for which convexity of the utility possibility set is a notion with interesting behavioral meaning.

lectures we will come to appreciate the power of these methods for studying problems of public decision-making.

One way of describing a Pareto optimum is to say that each Pareto optimum solves a constrained maximization problem where we fix Bruce's utility at some level and then maximize Anne's utility subject to the constraint that Bruce receives at least his assigned level of utility. We should, in principle, be able to generate the entire set of Pareto optimal situations by repeating this operation, fixing Bruce on different indifference levels.

Suppose that Anne's utility function is $U^A(C,T)$ and Bruce's utility function is $U^B(C,T)$. To find one Pareto optimum, pick a level of utility U^B for Bruce and find (C,T) to maximize $U^A(C,T)$ subject to the constraint that $U^B(C,T) \geq U^B$. A convenient tool for the study of problems of maximization subject to constraints is the method of Lagrange multipliers. The fact that we need to know is the following:

Theorem 1 (Kuhn-Tucker Theorem) *Let $f(\cdot)$ and $g^1(\cdot), \cdots, g^k(\cdot)$ be differentiable real valued functions of n real variables. Then (subject to certain regularity conditions) a necessary condition for $\bar{x}$ to yield an interior maximum of $f(\cdot)$ subject to the constraints that $g^i(x) \leq 0$ for all i is that there exist real numbers $\lambda^1 \geq 0, \cdots, \lambda^k \geq 0$, such that the "Lagrangean" expression*

$$L(x, \lambda^1, \cdots, \lambda^k) \equiv f(x) - \sum_{j=1} \lambda^j g^j(x) \tag{1.1}$$

has each of its partial derivatives equal to zero at $\bar{x}$. Furthermore, it must be that for all j, either $\lambda^j = 0$ or $g^j(\bar{x}) = 0$.

Returning to Anne and Bruce; a Pareto optimum is found by finding $(\bar{C}, \bar{T})$ and λ such that the partial derivatives of the Lagrangean,

$$L(C, T, \lambda) = U^A(C,T) - \lambda[\bar{U}^B - U^B(\ C, B)] \tag{1.2}$$

with respect to C and T are both zero when $C = \bar{C}$ and $T = \bar{T}$.

This tells us that:

$$\frac{\partial U^A(\bar{C}, \bar{T})}{\partial C} + \lambda \frac{\partial U^B(\bar{C}, \bar{T})}{\partial C} = 0 \tag{1.3}$$

$$\frac{\partial U^A(\bar{C}, \bar{T})}{\partial T} + \lambda \frac{\partial U^B(\bar{C}, \bar{T})}{\partial T} = 0 \tag{1.4}$$

Recall also that we must have $\lambda \geq 0$. Therefore from Equations 1.3 and 1.4 we see that at a Pareto optimum the marginal utilities of cribbage for

Anne must be of the opposite sign from the marginal utility of cribbage for Bruce. Likewise their marginal utilities for temperature must have opposite signs at a Pareto optimal point.

We can use the two equations 1.3 and 1.4 to eliminate the variable λ and we deduce that

$$\frac{\frac{\partial U^A(\bar{C},\bar{T})}{\partial C}}{\frac{\partial U^A(\bar{C},\bar{T})}{\partial T}} = \frac{\frac{\partial U^B(\bar{C},\bar{T})}{\partial C}}{\frac{\partial U^B(\bar{C},\bar{T})}{\partial T}}. \tag{4}$$

Thus we see that Anne's marginal rate of substitution between cribbage and temperature must be the same as Bruce's at any Pareto optimal point.

Notice that the term $\bar{U}^B$ does not enter equation (4). This condition must hold regardless of the level, $\bar{U}^B$, at which we set Bruce's utility. In general there will be many solutions of (4) corresponding to different points on the locus of Pareto optimal points in Figure 1.1 or equivalently to different levels of $\bar{U}^B$.

Using the Kuhn-Tucker method, we have uncovered all of the optimality conditions that we saw from the diagram. Since we already knew the answer, this may not seem like a big gain. But what we will soon discover is that we now have a tool that will sometimes enable us to analyze cases that are much too complicated for graphs.

Incidentally, if we want to find necessary conditions that a point is on the boundary of the utility possibility set, though not necessarily on the utility possibility frontier, the mathematical problem that we pose is a little different. For any choice of utility for Bruce, U^B, we choose (C, T) to maximize Anne's utility subject to the constraint that $U^B(C, T)$ *equals* U^B instead of being at least as large as U^B. This means that we simply apply the theory of Lagrange multipliers in the usual way, by looking for a critical point of

$$L(x, \lambda^1, \cdots, \lambda^k) \equiv f(x) - \sum_{j=1} \lambda^j g^j(x). \tag{1.5}$$

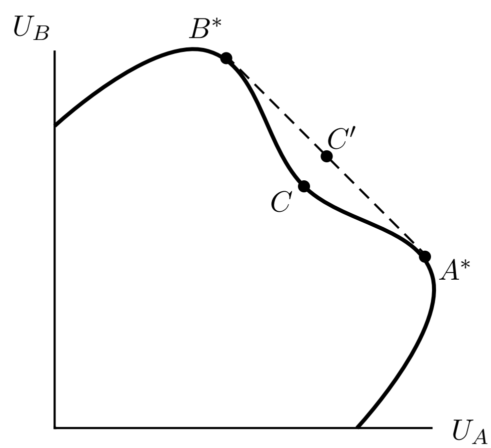
When we do this, we find precisely the same equations 1.3 and 1.4 that we found while applying the Kuhn-Tucker theorem. The only difference is that the restriction that $\lambda \geq 0$ does not apply. This is consistent with our earlier observation in the case of Anne and Bruce. The non Pareto optimal point V where indifference curves are tangent is a point on the southeast boundary of the utility possibility set but not on the utility possibility frontier. It satisfies the Lagrange multiplier conditions, but not the Kuhn-Tucker condition that $\lambda \geq 0$.

Gambles and utility

Analysis of the utility possibility frontier suggests some interesting possibilities for random allocations. Suppose that

Anne and Bruce are both von Neuman-Morgenstern, expected utility maximizers. Then there exists a utility function $U_A(C_A, T_A)$ such that Anne's preferences over gambles in which she has probability π of experiencing the situation (C_A, T_A) and probability $1-\pi$ of experiencing the situation (C'_A, T'_A) are represented by the expected utility function $E(U_A(C_A, T_A) = \pi U_A(C_A, T_A) + (1-\pi)U_A(C'_A, T'_A)$. Similarly Bruce's preferences are represented by an expected utility function of the form $E(U_B(C_B, T_B))$. As you may recall, von Neuman-Morgenstern representations of utility are unique only up to monotonic increasing affine transformations (that is, multiplication by a positive number and addition of a constant).

Figure 1.3: Expected Utilities and a Lottery



Suppose you draw a utility possibility set corresponding to a von Neuman-Morgenstern representation of utility. Call this the *sure thing utility possibility set*. Now suppose this set is non-convex as in Figure 1.3. Consider a point like C in Figure 1.3 that is on an "inward bulge" of the sure thing utility possibility frontier. It is possible to arrange a gamble that gives both Anne and Bruce a higher utility than they would have by accepting the situation C with certainty. For example, suppose that a lottery is held in which with probability 1/2, the situation will be A^* and with probability 1/2, the situation will be B^*. Then the expected utilities of Anne and Bruce would be given by the point C'. At C', both have higher expected utilities than they would have if the outcome were C with certainty. If random choices were not possible, the point C would be Pareto optimal. But if lotteries are available, then C is not Pareto optimal, since a lottery in which outcomes A^* and B^* are equally likely is Pareto superior to a certainty of outcome C.

More generally, any point that is a convex combination of two points in the sure thing utility possibility set represents an expected utility distribution that can be achieved when the situation is chosen by lottery. The set generated in this way is what is known mathematically as the *convex hull* of the sure thing utility possibility set. If the sure thing utility possibility set is convex, then for any gamble, there is always some certainty situation which is at least as good for everybody. Therefore, no beneficial social purpose could be served by gambles. But if the utility possibility set is not convex, then it is possible to find lotteries that give everybody a higher expected utility than they would get with situations that lie out on the frontier of the sure thing utility possibility set. Some kind of random choice then needs to be introduced in order to exploit the full range of Pareto optimal possibilities.

Exercises

1.1 Suppose that Anne's preferences are represented by the utility function $U^A(C,T) = -[(C-20)^2+(T-25)^2]$ and Bruce's preferences are represented by the utility function $U^B(C,T) = -[(C-10)^2+(T-15)^2]$.

a). Sketch their indifference curves on a diagram.

b). Is the situation $(10, 15)$ Pareto optimal?

c). Find the set of all Pareto optimal situations.

> **Hint:** *While Lagrangean analysis solves this problem nicely, it could also be solved by plain plane geometry.*

d). Find the set of situations that is Pareto superior to $(9, 14)$.

e). Find a situation in which Anne's indifference curve is tangent to Bruce's but which is not Pareto optimal. Explain what is going on.

f). If situation α is Pareto optimal and situation β is not Pareto optimal, must α be Pareto superior to β? Explain.

1.2 Find and draw the utility possibility frontier for Anne and Bruce in the previous problem.

> **Hint:** *You know that all of the Pareto efficient allocations lie on a line. You should be able to trace out the distribution of utility as you move along this line.*

1.3 Suppose that the utility functions proposed in Problem 1.1 are the von Neuman-Morgenstern representations of both peoples' preferences.

a). Can they improve on any of the sure thing Pareto optimal allocations by gambling?

b). Suppose that the utility functions in problem 1.1 represent sure- thing preferences but in order to get the von Neuman-Morgenstern representation you have to make non-affine monotone increasing transformations of these utility functions. Give an example of a von Neuman-Morgenstern utility function that represents the same sure thing preferences as that in Problem 1 but where the answer to the question "Can you improve on sure thing Pareto optimal allocations by gambling?" is different from the answer you gave in Part a.

1.4 In Figure 1.1, the "elliptical" indifference curves that are drawn are "sloped" rather than having their major and minor axes parallel to the vertical and horizontal axes. What does this signify about Anne's and Bruce's preferences?

1.5 In Figure 1.2 the boundary of the utility possibility set slopes uphill to the west of point B. Explain why this is so.

1.6 Suppose that Anne and Bruce lose interest in cribbage, but are still concerned about two things ... the room temperature and breakfast time. In principle, breakfast could be served at any time on a 24 hour clock.

a). On the tube from the center of a roll of toilet paper, or some similar material draw indifference curves for Anne and Bruce. (Use two colors of ink). Show the locus Pareto optimal situations.

b). Suppose the only two issues of concern are breakfast time and dinner time. Explain how you could use a bagel (or a doughnut if you are Canadian) to diagram indifference curves and Pareto optimal points. What can you tell us about the set of Pareto optimal points?

1.7 Suppose that Charlie moves in with Anne and Bruce. The three of them learn three–handed cribbage and lose all interest in two–handed cribbage. Anne's and Bruce's utilities are as before, while Charlie's utility function is:

$$U^C(C,T) = -[(C-20)^2 + (T-15)^2]. \tag{1.6}$$

Find the new set of Pareto optimal situations.

> **Hint:** *It is easier to understand what is going on in this problem if you graph the situation and stare hard at it than if you go at with brute force Lagrangeans.*[4]

After you have seen the graphical solution, show that your answer is consistent with the Kuhn-Tucker method. What is it about this problem that makes the Kuhn-Tucker solution work so "oddly"?

[4]This is sound advice in general.