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The Non-Constant Sum Colonel Blotto Game

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Abstract The Colonel Blotto game is a two-player constant-sum game in which each player simultaneously distributes her fixed level of resources across a set of contests. In the traditional formulation of the Colonel Blotto game, the players' resources are "use it or lose it" in the sense that any resources which are not allocated to one of the contests are forfeited. This paper examines a non-constant-sum version of the Colonel Blotto game which relaxes this use it or lose it feature. We find that if the level of asymmetry between the players' budgets is below a threshold, then the unique set of equilibrium univariate marginal distributions of the non-constant-sum game is equivalent up to an affine transformation to the unique set of equilibrium univariate marginal distributions of the constant-sum game. Once the asymmetry of the players' budgets exceeds the threshold we construct a new equilibrium.

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1 Introduction

Kvasov (2007) introduces a non-constant-sum version of the classic Colonel Blotto game. Originating with Borel (1921), the Colonel Blotto game examines strategic resource allocation across multiple simultaneous contests. Borel formulates this problem as a constant-sum game involving two players, A and B, who must each allocate a fixed amount of resources, $X_A = X_B$, over a finite number of contests. Each player must distribute their resources without knowing their opponent's distribution of resources. In each contest, the player who allocates the higher level of resources wins, and the payoff for the whole game is the sum of the wins across the individual contests. A novel feature of the Colonel Blotto game is that a mixed strategy is a multivariate distribution function in which each individual contest is represented as a dimension. The restriction on the players' expenditures implicitly places a constraint on the support of the players' joint distributions. Namely, each point contained in the support of a player's joint distribution must satisfy their budget constraint with probability one.

While a focal point in the early game theory literature,[1] the Colonel Blotto game has also experienced a recent resurgence of interest (see for example Golman and Page (2006), Hart (2008), Kovenock and Roberson (2007), Laslier (2002), Laslier and Picard (2002), Roberson (2008), or Weinstein (2005)). Most closely related to this paper are Roberson (2006) and Kvasov (2007). For all configurations of the asymmetric Colonel Blotto game with three or more contests, Roberson (2006) provides the characterization of the unique equilibrium payoffs.[2] The characterization of the equilibrium univariate marginal distributions and the existence of joint distributions which provide the equilibrium univariate marginal distributions and expend the players' respective budgets with probability one are also given in Roberson (2006).

In Borel's original formulation of the Colonel Blotto game the players' resources are "use it or lose it" in the sense that any resources which are not allocated to one of the contests are forfeited. Kvasov's (2007) non-constant-sum version of the Colonel Blotto game relaxes this use it or lose it feature. In the case of symmetric budgets, that paper

[1] See Kvasov (2007) or Roberson (2006) for surveys of this literature.

[2] The case of $n = 2$, with symmetric and asymmetric forces, is discussed by Gross and Wagner (1950). Moving from $n = 2$ to $n \geq 3$ greatly enlarges the space of feasible n-variate distribution functions, and the equilibrium strategies examined in that paper differ dramatically from the case of $n = 2$.

establishes that a suitable affine transformation of the constant-sum equilibrium is an equilibrium of the non-constant-sum game.

In this paper we extend the analysis of the non-constant-sum version of the Colonel Blotto game to allow for asymmetric budget constraints. As long as the level of asymmetry between the players' budgets is below a threshold, we find that there exists an affine transformation of the equilibrium to the constant-sum game which provides an equilibrium to the non-constant-sum game. Once the asymmetry of the players' budgets exceeds the threshold this correspondence breaks down and we construct an entirely new equilibrium. For all configurations of the players' aggregate levels of force we characterize the unique equilibrium payoffs, and for most parameter configurations we characterize the complete set of equilibrium univariate marginal distributions.

Section 2 presents the model. Section 3 characterizes the equilibrium payoffs and the equilibrium set of univariate marginal distributions for the asymmetric non-constant-sum version of the Colonel Blotto game. Section 4 concludes.

2 The Model

Two players, A and B, simultaneously enter bids in a finite number, $n \geq 3$, of independent all-pay auctions. Each contest has a common value of v for each player. Each player has a fixed level of available resources (or budget), X_i for $i = A, B$. Let $X_A \leq X_B$, and define the modified budgets as $\overline{X}_A = \min\{X_A, nv/2\}$ and $\overline{X}_B = \min\{X_B, \sqrt{nv\overline{X}_A/2}\}$.[3] In the case that the players enter the same bid in a given contest, it is assumed that player B wins the auction if the common bid is X_A, otherwise each player wins the auction with equal probability. The specification of the tie-breaking rule does not affect the results as long as $(2/n)\overline{X}_B \leq \overline{X}_A$. In the case that $(2/n)\overline{X}_B > \overline{X}_A$, this specification of the tie-breaking rule avoids the need to have player B provide a bid arbitrarily close to, but above, player A's maximal bid, X_A. A range of tie-breaking rules yield similar results.

[3] As shown in Appendix A, $\overline{X}_i$ corresponds to the equilibrium expected expenditure for player i. This specification of $\overline{X}_i$ allows for a unified treatment of the three possible cases: (a) neither player using all of her available resources, (b) only the weaker player (A) using all of her available resources, and (c) both players A and B using all of their available resources.

Each contest is modeled as an all-pay auction. The payoff to player i for a bid of b_i^j in contest j is given by

$$\pi_i^j = \begin{cases} v - b_i^j & \text{if } b_i^j > b_{-i}^j \\ -b_i^j & \text{if } b_i^j < b_{-i}^j \end{cases}$$

where ties are handled as described above. Each player's payoff across all n all-pay auctions is the sum of the payoffs across the individual auctions.

The bid provided to each all-pay auction must be nonnegative. For player i, the set of feasible bids across the n all-pay auctions is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{b} \in \mathbb{R}_+^n \middle| \sum_{j=1}^{n} b_i^j \leq X_i \right\}.$$

It will also be useful to define the set of n-tuples which exhaust the modified budgets $\overline{X}_A$ and $\overline{X}_B$. Let $\overline{\mathfrak{B}}_i$ denote this set, defined as

$$\overline{\mathfrak{B}}_i = \left\{ \mathbf{b} \in \mathbb{R}_+^n \middle| \sum_{j=1}^{n} b_i^j = \overline{X}_i \right\}.$$

Strategies

It is well known that there are no pure strategy equilibria for this class of games. A mixed strategy, which we term a *distribution of resources*, for player i is an n-variate distribution function $P_i : \mathbb{R}_+^n \to [0,1]$ with support (denoted $Supp(P_i)$) contained in the set of player i's set of feasible bids $\mathfrak{B}_i$ and with one-dimensional marginal distribution functions $\{F_i^j\}_{j=1}^n$, one univariate marginal distribution function for each all-pay auction j. The n-tuple of player i's bids across the n all-pay auctions is a random n-tuple drawn from the n-variate distribution function P_i.

The Non-Constant-Sum Colonel Blotto game

The N-C-S Colonel Blotto game, which we label

$$NCB\{X_A, X_B, n, v\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of resources subject to their budget constraints, each all-pay auction is won by the player that provides the higher bid in that auction (where in the case of a tie the tie-breaking rule described above applies), and players' receive the sum of their payoffs across all of the all-pay auctions.

Note that in the non-constant-sum Colonel Blotto game two players simultaneously compete in a set of independent all-pay auctions subject to their respective budget constraints. The presence of the budget constraints gives rise to strategic considerations which are reminiscent of those arising in the single all-pay auction with budget-constrained bidders (see Che and Gale (1998)). However, in the non-constant-sum Colonel Blotto game the budget constraints hold not within one auction but across the entire set of auctions. As will be seen, the equilibria of these two games differ in fundamental ways.

Before proceeding with the analysis, it is also instructive to compare this formulation with that of the constant-sum Colonel Blotto game. The constant-sum Colonel Blotto game differs from the non-constant-sum game in that in each contest j the payoff to each player i for a bid of b_i^j is given by

$$\pi_i^j = \begin{cases} 1 & \text{if } b_i^j > b_{-i}^j \\ 0 & \text{if } b_i^j < b_{-i}^j \end{cases}$$

where ties are handled as described above. Note that, in the constant-sum game resources which are not allocated to one of the contests have no value; that is, resources are use it or lose it. Each player's payoff across all n contests is the sum of the wins across the contests to which the player provides a higher bid.

3 Optimal Distributions of Resources

The following four theorems examine the equilibrium distributions of resources for all symmetric and asymmetric configurations of resource levels. Theorems 1, 2 and 4 characterize the unique sets of equilibrium univariate marginal distributions and the unique equilibrium payoffs. Theorem 3 provides the unique equilibrium payoffs and a pair of equilibrium distributions of resources.[4]

The first two theorems address the portion of the parameter space in which there exists an affine transformation (with respect to the modified budgets) of the equilibrium of the constant-sum game which constitutes an equilibrium of the non-constant-sum game. Once $(\overline{X}_A/\overline{X}_B) < (2/n)$ and $X_B > (n-1)X_A$ the correspondence between these

[4] In this parameter range there exist a continuum of equilibrium univariate marginal distributions.

two games breaks down. Theorem 3 is based on the equilibrium of the constant-sum game. However, in this case the transformation entails a more involved modification to the support of the distribution. We conclude with Theorem 4 which constructs entirely new equilibrium distributions of resources in the remaining parameter range.

For the game $NCB\{X_A, X_B, n, v\}$, Theorem 1 examines all configurations of resource levels X_A and X_B which satisfy $(2/n) < (\overline{X}_A/\overline{X}_B) \leq 1$.

Theorem 1 *Let X_A, X_B, v, and $n \geq 3$ satisfy $(2/n) \leq (\overline{X}_A/\overline{X}_B) \leq 1$. The pair of n-variate distribution functions P_A^* and P_B^* constitute a Nash equilibrium of the game $NCB\{X_A, X_B, n, v\}$ if and only if they satisfy the two conditions: (1) $Supp(P_i^*) \subset \mathfrak{B}_i$ and (2) P_i provides the corresponding set of univariate marginal distribution functions $\{F_i^j\}_{j=1}^n$ outlined below.*

For player A the unique set of equilibrium univariate marginal distributions $\{F_A^j\}_{j=1}^n$ are described as follows

$$\forall\ j \in \{1, \ldots, n\} \quad F_A^j(x) = \left(1 - \frac{\overline{X}_A}{\overline{X}_B}\right) + \frac{x}{(2/n)\overline{X}_B}\left(\frac{\overline{X}_A}{\overline{X}_B}\right) \quad \text{for } x \in \left[0, \frac{2}{n}\overline{X}_B\right].$$

Similarly for player B

$$\forall\ j \in \{1, \ldots, n\} \quad F_B^j(x) = \frac{x}{(2/n)\overline{X}_B} \quad \text{for } x \in \left[0, \frac{2}{n}\overline{X}_B\right].$$

The unique equilibrium expected payoff for player A is $(nv\overline{X}_A/2\overline{X}_B) - \overline{X}_A$, and the unique equilibrium expected payoff for player B is $nv(1 - (\overline{X}_A/2\overline{X}_B)) - \overline{X}_B$.

The existence of a pair of n-variate distribution functions which satisfy conditions (1) and (2) of Theorem 1 is provided in Roberson (2006). In particular, Roberson (2006) establishes the existence of n-variate distribution functions for which $Supp(P_i^*) \subset \overline{\mathfrak{B}}_i$ and that provide the necessary sets of univariate marginal distribution functions given in Theorem 1. The proof of uniqueness of the univariate marginal distribution functions and equilibrium payoffs is given in Appendix A.

An important distinction between the constant-sum and the non-constant-sum versions of the game is that in the constant-sum version each player expends all of her resources with probability one as long as $(1/n-1) \leq (X_A/X_B) \leq 1$. This need not be the case in the non-constant-sum game. In particular there are three possible cases: (a) neither player uses all of her available resources, (b) only (the weaker) player A uses all of her available resources, and (c) both players A and B use all of their available resources.

While it is straightforward to show that any pair of n-variate distribution functions which satisfy conditions (1) and (2) of Theorem 1 form an equilibrium, it is useful to provide the intuition for this result. We begin with the equilibrium expected payoffs for each player and any X_A and X_B contained in the portion of the parameter space for which Theorem 1 applies, and then examine these payoffs in each of the three possible cases. Let P_B^* denote a feasible n-variate distribution function for player B with the univariate marginal distributions $\{F_B^j\}_{j=1}^n$ given in Theroem 1. If player B is using P_B^*, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{b}_A \in \mathfrak{B}_A$ (one bid for each of the n all-pay auctions) such that $b_A^j \in [0,(2/n)\overline{X}_B]$ for each auction j, is

$$\pi_A(\mathbf{b}_A, P_B^*) = \sum_{j=1}^{n} \left[vF_B^j\left(b_A^j\right) - b_A^j \right].$$

Recall that for all j, $F_B^j(x) = \frac{x}{(2/n)\overline{X}_B}$ for $x \in [0,(2/n)\overline{X}_B]$. Simplifying yields

$$\pi_A(\mathbf{b}_A, P_B^*) = \left(\frac{nv}{2\overline{X}_B} - 1\right) \sum_{j=1}^{n} b_A^j. \tag{1}$$

The expected payoff π_B to player B from any n-tuple of bids across the n all-pay auctions $\mathbf{b}_B \in \mathfrak{B}_B$ such that $b_B^j \in (0,(2/n)\overline{X}_B]$ for each auction j — when player A uses a feasible n-variate distribution P_A^* with the univariate marginal distributions $\{F_A^j\}_{j=1}^n$ given in Theroem 1 — follows directly,

$$\pi_B(\mathbf{b}_B, P_A^*) = nv\left(1 - \frac{\overline{X}_A}{\overline{X}_B}\right) + \left(\frac{nv\overline{X}_A}{2\overline{X}_B^2} - 1\right) \sum_{j=1}^{n} b_B^j. \tag{2}$$

Observe that neither player can bid below 0 and that bidding above $(2/n)\overline{X}_B$ is suboptimal. Thus, (1) and (2) provide the maximal payoffs (for player A and player B respectively) for any feasible n-tuple of bids across the n all-pay auctions.

Suppose that we are in case (a) in which neither player uses all of her available resources. Case (a) corresponds to the situation in which the total value of the n auctions nv is low enough relative to the players' budgets that neither player has incentive to commit all of her resources. If player A does not use all of her budget, then from $\overline{X}_A = \min\{X_A, nv/2\}$ it must be that $X_A > (nv/2)$ and so $\overline{X}_A = (nv/2)$. Similarly from $\overline{X}_B = \min\{X_B, \sqrt{nv\overline{X}_A/2}\}$, it follows that if player A (the weaker player) is not using all of her budget then $\overline{X}_B = (nv/2)$. Given that $\overline{X}_A = \overline{X}_B = (nv/2)$, the expected payoffs given in (1) and (2) are $\pi_A(\mathbf{b}_A, P_B^*) = 0$ and $\pi_B(\mathbf{b}_B, P_A^*) = 0$ respectively. Observe that in case (a) neither player has incentive to change the aggregate level of resources that they commit to the n all-pay auctions. That is, given that the opponent is using the equilibrium strategy, the expected payoff to each player is independent of the aggregate level of resources that they commit across the n all-pay auctions.

Now suppose that we are in case (b) in which only player A uses all of her budget. Case (b) corresponds to the situation in which the total value of the n all-pay auctions nv is high enough that the weaker player optimally commits all of her resources but not so high that the stronger player must also commit all of her resources to the n all-pay auctions. From the proceeding discussion it follows that $X_A \leq (nv/2)$ and thus $\overline{X}_A = X_A$. If player B is not using all of her budget then from $\overline{X}_B = \min\{X_B, \sqrt{nvX_A/2}\}$, it must be that $X_B > \sqrt{nvX_A/2}$ and so $\overline{X}_B = \sqrt{nvX_A/2}$. Inserting $\overline{X}_A$ and $\overline{X}_B$ into (1) and (2) and simplifying yields

$$\pi_A(\mathbf{b}_A, P_B^*) = \left(\sqrt{\frac{nv}{2X_A}} - 1\right) \sum_{j=1}^{n} b_A^j \tag{3}$$

and

$$\pi_B(\mathbf{b}_B, P_A^*) = nv\left(1 - \sqrt{\frac{2X_A}{nv}}\right). \tag{4}$$

Recall that in case (b) $X_A \leq (nv/2)$ and so $(\sqrt{nv/2X_A} - 1) \geq 0$. From (3) we see that player A is indifferent with regards to which all-pay auctions to commit resources to, but has incentive to increase her aggregate level of resource commitment across the n all-pay auctions. However in case (b), player A is at her budget constraint and her equilibrium distribution of resources P_A^* expends her budget with probability one.[5] From (4) we see that the expected payoff to player B is independent of the aggregate level of resources that she commits across the n all-pay auctions (so long as she commits a strictly positive level of resources to each auction), and so player B does not have incentive to change the aggregate level of resources that she commits to the n all-pay auctions.

Finally, suppose that we are in case (c) in which both players use all of their budgets. Case (c) corresponds to the situation in which the total value of the n all-pay auctions nv is high enough that both players optimally commit all of their resources to the n all-pay auctions. Thus, $\overline{X}_A = X_A$ and $\overline{X}_B = X_B$. From (1) and (2) it follows that

$$\pi_A(\mathbf{b}_A, P_B^*) = \left(\frac{nv}{2X_B} - 1\right) \sum_{j=1}^{n} b_A^j \tag{5}$$

and

$$\pi_B(\mathbf{b}_B, P_A^*) = nv\left(1 - \frac{X_A}{X_B}\right) + \left(\frac{nvX_A}{2X_B^2} - 1\right) \sum_{j=1}^{n} b_B^j. \tag{6}$$

[5] Recall that Roberson (2006) establishes the existence of n-variate distribution functions for which $Supp(P_i^*) \subset \overline{\mathfrak{B}}_i$, and that in this case $\overline{X}_A = X_A$. It follows directly that player A expends her budget with probability one.

In case (c), $X_A < (nv/2)$ and $X_B < \sqrt{nvX_A/2} < (nv/2)$. Observe in (5) that $((nv/2X_B) - 1) > 0$ and, thus, player A has incentive to increase her aggregate level of resource commitment across the n all-pay auctions, but in her equilibrium distribution of resources P_A^* she is already at her budget constraint with probability one. Similarly, in (6) $((nvX_A/2X_B^2) - 1) > 0$ and, thus, player B has incentive to increase her aggregate level of resource commitment across the n all-pay auctions, but in her equilibrium distribution of resources P_B^* she is already at her budget constraint with probability one.

Given that Roberson (2006) demonstrates the existence of a pair of n-variate distributions that satisfy conditions (1) and (2) of Theorem 1, it follows from the arguments given above that such a pair of n-variate distribution functions constitute an equilibrium in all three cases (a), (b), and (c). The proof of the uniqueness of the sets of univariate marginal distributions is given in Appendix A.

The following Theorem addresses the remaining portion of the parameter space for which there exists an affine transformation of the equilibrium of the constant-sum game which constitutes an equilibrium of the non-constant-sum game.

Theorem 2 *Let X_A, X_B, v, and $n \geq 3$ satisfy $(\overline{X}_A/\overline{X}_B) < (2/n)$ and $X_B \leq (n-1)X_A$. The pair of n-variate distribution functions P_A^* and P_B^* constitute a Nash equilibrium of the game $NCB\{X_A, X_B, n, v\}$ if and only if they satisfy the two conditions: (1) $Supp(P_i^*) \subset \mathfrak{B}_i$ and (2) P_i provides the corresponding set of univariate marginal distribution functions $\{F_i^j\}_{j=1}^n$ outlined below.*

For player A the unique set of equilibrium univariate marginal distribution functions $\{F_A^j\}_{j=1}^n$ are described as follows

$$\forall\ j \in \{1,\ldots,n\} \quad F_A^j(x) = \left(1 - \tfrac{2}{n}\right) + \tfrac{x}{X_A}\left(\tfrac{2}{n}\right) \quad \text{for } x \in [0, X_A].$$

Similarly for player B

$$\forall\ j \in \{1,\ldots,n\} \quad F_B^j(x) = \begin{cases} \frac{2x\left(X_A - \frac{X_B}{n}\right)}{(X_A)^2} & \text{for } x \in [0, X_A) \\ 1 & \text{for } x \geq X_A \end{cases}.$$

The unique equilibrium expected payoff for player A is $nv((2/n) - ((2X_B)/(n^2X_A))) - X_A$, and the unique equilibrium expected payoff for player B is $nv(1-(2/n)) + nv((2X_B)/(n^2X_A)) - X_B$.

The existence of a pair of n-variate distribution functions which satisfy conditions (1) and (2) of Theorem 2 is provided in Roberson (2006). The proof of uniqueness of the univariate marginal distributions and equilibrium payoffs is given in Appendix A.

Before proceeding with a sketch of the proof that a pair of n-variate distributions that satisfy conditions (1) and (2) of Theorem 2 form an equilibrium, it is helpful to trace out the Theorem 2 parameter range. Since $\overline{X}_B = \min\{X_B, \sqrt{nv\overline{X}_A/2}\}$ and $(\overline{X}_A/\overline{X}_B) < (2/n)$ it follows that

$$\overline{X}_A < \frac{2}{n}\overline{X}_B \leq \sqrt{\frac{2v\overline{X}_A}{n}}$$

and so $\overline{X}_A < (2v/n)$. Therefore it must be the case that $\overline{X}_A = X_A$. It also follows that $X_B \leq (n-1)X_A$ combined with $(n-1)X_A < \sqrt{nvX_A/2}$ implies that $\overline{X}_B = X_B$. Thus, the Theorem 2 parameter range is given by $0 \leq X_A < (2v/n)$ and $(n/2)X_A < X_B \leq (n-1)X_A$

Returning to the sketch of the proof that a pair of n-variate distributions that satisfy conditions (1) and (2) of Theorem 2 form an equilibrium, let P_B^* denote a feasible n-variate distribution for player B with the univariate marginal distributions $\{F_B^j\}_{j=1}^n$ given in Theorem 2. If player B is using P_B^*, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{b}_A \in \mathfrak{B}_A$ such that $b_A^j \in [0, X_A)$ for each auction j, is

$$\pi_A(\mathbf{b}_A, P_B^*) = \left(\frac{2v(X_A - (X_B/n))}{X_A^2} - 1\right)\sum_{j=1}^n b_A^j. \tag{7}$$

Note that $(2v/X_A^2)(X_A - (X_B/n)) - 1 \geq 0$ is equivalent to $X_B \leq (n - (nX_A/2v))X_A$. Since $X_A < (2v/n)$, it follows from (7) that player A has incentive to expend all of her available resources in the n all-pay auctions not only in expectation but with certainty.

Similarly, the expected payoff π_B to player B from any n-tuple of bids across the n all-pay auctions $\mathbf{b}_B \in \mathfrak{B}_B$ such that $b_B^j \in (0, X_A]$ for each auction j, when player A uses a feasible n-variate distribution P_A^* with the univariate marginal distributions $\{F_A^j\}_{j=1}^n$ given in Theroem 2, is

$$\pi_B(\mathbf{b}_B, P_A^*) = nv\left(1 - \frac{2}{n}\right) + \left(\frac{2v}{nX_A} - 1\right)\sum_{j=1}^n b_B^j. \tag{8}$$

Since $X_A < (2v/n)$ it follows that $(2v/nX_A) - 1 > 0$, and, thus, player B has incentive to expend all of her available resources in the n all-pay auctions with certainty.

Given that Roberson (2006) demonstrates the existence of a pair of n-variate distributions that result in the sets of univariate marginal distributions given in Theorem 2 and that satisfy the budget restriction with probability 1, it follows from the arguments given above that such a pair of n-variate distribution functions constitute an equilibrium. The proof of uniqueness of the univariate marginal distributions is given in Appendix A.

While the first two theorems involve affine transformations (with respect to the modified budgets) of the equilibrium of the constant-sum game, once $(\overline{X}_A/\overline{X}_B) < (2/n)$ and $X_B > (n-1)X_A$ the correspondence between the constant-sum and non-constant-sum games breaks down. For the remaining parameter range, Theorems 3 and 4 construct new equilibrium joint distributions. Theorem 3, which addresses the case that $(\overline{X}_A/\overline{X}_B) > (2/n)$ and $\min\{nX_A, (n-2)X_A + (2v/n)\} > X_B > (n-1)X_A$, is also based on the equilibrium of the constant-sum game but includes a more involved modification of the support. Theorem 4, which addresses the remaining case that $(\overline{X}_A/\overline{X}_B) < (2/n)$ and $X_B \geq \min\{nX_A, (n-2)X_A + (2v/n)\}$ (note that if $X_A < (2v/n)$ then $\min\{nX_A, (n-2)X_A + (2v/n)\} > (n-1)X_A$), constructs entirely new equilibrium distributions of resources.

Before turning to the statements of Theorems 3 and 4, observe that while the relationship between the constant-sum and and non-constant-sum versions of the game is linear with respect to the modified budgets — as long as the level of asymmetry between the players' budgets is below the threshold given in Theorem 2 — the relationship between these games with respect to the aggregate resource levels is highly non-linear. Panel (i) of Figure 1 illustrates the regions of the parameter space corresponding to each of the four theorems in the non-constant-sum game, and Panel (ii) of Figure 1 illustrates the regions which correspond, for the constant-sum game, to Theorems 2, 3, and 5 of Roberson (2006).

[Insert Figure 1 here]

In the constant-sum game, four rays emanating from the origin partition the parameter space into four disjoint regions. As shown in Panel (ii) of Figure 1, these regions are delineated by (1) $X_A = (X_B/(n-1))$, (2) $X_A = (2X_B/n)$, and (3) $X_A = X_B$. While Theorems 1, 2 and 3 of this paper (the non-constant-sum game) are transformations of Theorems 2, 3 and 5 of Roberson (2006) (the constant-sum game) respectively, the corresponding parameter regions differ in nontrivial ways. The complicating factor in the relationship between the two versions of the game is the strategic considerations arising from the use it or lose it feature of the constant-sum formulation and the corresponding relaxation of this feature in the non-constant-sum formulation. In particular, recall that in the non-constant-sum game with resource levels which satisfy $(2/n) \leq (\overline{X}_A/\overline{X}_B) \leq 1$ (as in Theorem 1) there were three possible cases: (a) neither player uses all of their available resources, (b) only (the weaker) player A uses all of her available resources, and (c) both players A and B use all of their available resources. (The regions corresponding to each of these cases is labeled in panel (i) of Figure 1.)

Furthermore, in the region in which $X_A < (X_B/n)$ the constant-sum game is trivial since resources are use it or lose it and the stronger player (B) has a sufficient level of resources to win each of the n contests with certainty. In this is region there is no relationship between the two games. Due to the relaxation of the use it or lose it feature, the non-constant-sum game never becomes trivial, and for the non-constant-sum game Theorem 4 constructs entirely new equilibrium distributions of resources in the remaining parameter range.

In the case that $(\overline{X}_A/\overline{X}_B) > (2/n)$ and $\min\{nX_A, (n-2)X_A + (2v/n)\} > X_B > (n-1)X_A$ Theorem 2 would provide the unique set of equilibrium marginal distributions if a sufficient n-variate distribution function were to exist for each player. As in the corresponding constant-sum parameter range, such a joint distribution fails to exist for player B. One equilibrium is given by an extension of the case of $n = 2$ with asymmetric forces discussed by Gross and Wagner (1950). The set of equilibrium univariate marginal distributions is not unique, but the equilibrium payoffs are unique.

Theorem 3 *Define* $k = \lceil (X_A)/(X_B - X_A(n-1)) \rceil$. *Let* X_A, X_B, v, *and* $n \geq 3$ *satisfy* $(\overline{X}_A/\overline{X}_B) > (2/n)$ *and* $\min\{nX_A, (n-2)X_A + (2v/n)\} > X_B > (n-1)X_A$. *A Nash equilibrium of the game NCB*$\{X_A, X_B, v, n\}$ *is for each player to allocate her resources according to the following n-variate distributions:*

Player A randomly allocates 0 *resources to* $n-2$ *of the all-pay auctions, each all-pay auction chosen with equal probability,* $(n-2)/n$. *On the remaining* 2 *all-pay auctions player A utilizes a bivariate distribution function with k mass points, each mass point receiving the same weight,* $(1-(nX_A)/(2v))/k$. *Player A's mass points on these two remaining all-pay auctions are located at the points*

$$\left((k-1-i)\,\frac{X_A}{k-1}, i\frac{X_A}{k-1}\right),\ i = 0, \ldots, k-1.$$

Player A uniformly distributes the remaining $(nX_A)/(2v)$ *of the mass along her budget line* $\{(x_1, x_2) |\ x_1 + x_2 = X_A\}$.

Player B randomly allocates X_A *forces to* $n-2$ *all-pay auctions, each all-pay auction chosen with equal probability,* $(n-2)/n$. *On the remaining* 2 *all-pay auctions player B utilizes a bivariate distribution function with k mass points, each mass point receiving the same weight,* $(1-n(X_B - X_A(n-2))/(2v))/(k)$. *Player B's mass points on the* 2

remaining battlefields are located at

$$\left(X_A - i\frac{nX_A - X_B}{k-1}, X_A - (k-1-i)\frac{nX_A - X_B}{k-1}\right),\ i = 0, \ldots, k-1.$$

Player B uniformly distributes the remaining $(X_B - X_A(n-2))/(2v)$ *of the mass along her budget line* $\{(x_1,x_2)|\ x_1 + x_2 = X_B - X_A(n-2)\}$ *and the two line segments* $\{(x_1,x_2)|\ x_1 = X_A \text{ and } 0 \leq x_2 \leq X_B - X_A(n-1)\}$*, and* $\{(x_1,x_2)|\ x_2 = X_A \text{ and } 0 \leq x_1 \leq X_B - X_A(n-1)\}$*.*

The unique equilibrium expected payoff for player A is $v(k-1)\big((2/n) - ((X_B - (X_A(n-2))/v)\big)/k$*, and the unique equilibrium expected payoff for player B is* $(v - X_A)(n-2) + ((v2(n-2))/n) + v(k-1)\big((2/n) - (X_A/v)\big)/k$*.*

The proof of Theorem 3 is given in Appendix B.

The following Theorem constructs entirely new equilibrium distributions of resources for the portion of the parameter space in which the correspondence between the constant-sum and non-constant-sum versions of the game breaks down.

Theorem 4 *Let* X_A*,* X_B*,* v*, and* $n \geq 3$ *satisfy* $(\overline{X}_A/\overline{X}_B) < (2/n)$ *and* $X_B \geq \min\{nX_A, (n-2)X_A + (2v/n)\}$*. The pair of* n*-variate distribution functions* P_A^* *and* P_B^* *constitute a Nash equilibrium of the game NCB*$\{X_A, X_B, n, v\}$ *if and only if they satisfy the two conditions: (1)* $Supp(P_i^*) \subset \mathfrak{B}_i$ *and (2)* P_i *provides the corresponding set of univariate marginal distribution functions* $\{F_i^j\}_{j=1}^n$ *outlined below.*

For player A the unique set of equilibrium univariate marginal distribution functions $\{F_A^j\}_{j=1}^n$ *are described as follows*

$$\forall\ j \in \{1, \ldots, n\} \quad F_A^j(x) = \left(1 - \tfrac{X_A}{v}\right) + \tfrac{x}{v} \quad \text{for } x \in [0, X_A].$$

Similarly for player B

$$\forall\ j \in \{1, \ldots, n\} \quad F_B^j(x) = \begin{cases} \frac{x}{v} & \text{for } x \in [0, X_A) \\ 1 & \text{for } x \geq X_A \end{cases}.$$

The unique equilibrium expected payoff for player A is 0*, and the unique equilibrium expected payoff for player B is* $nv(1 - (X_A/v))$*.*

The existence of n-variate distributions which satisfy conditions (1) and (2) of Theorem 4 is provided in Appendix C. The proof of uniqueness of the univariate marginal distributions and equilibrium payoffs is given in Appendix A.

To see that these two sets of univariate marginal distributions form an equilibrium in the Theorem 4 parameter region, let P_B^* denote a feasible n-variate distribution for player

B with the univariate marginal distributions $\{F_B^j\}_{j=1}^n$ given in Theorem 4. If player B is using P_B^*, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{b}_A \in \mathfrak{B}_A$ is

$$\pi_A(\mathbf{b}_A, P_B^*) = 0. \tag{9}$$

From (9), player A does not have incentive to increase or decrease her level of resource commitment in the n all-pay auctions.

Similarly, the expected payoff π_B to player B from any n-tuple of bids across the n all-pay auctions $\mathbf{b}_B \in \mathfrak{B}_B$ such that $b_B^j \in (0, X_A]$ for each auction j, when player A uses a feasible n-variate distribution P_A^* with the univariate marginal distributions $\{F_A^j\}_{j=1}^n$ given in Theroem 4, is

$$\pi_B(\mathbf{b}_B, P_A^*) = nv\left(1 - \frac{X_A}{v}\right). \tag{10}$$

Thus, player B also has no incentive to increase or decrease her level of resource commitment in the n all-pay auctions.

Given that Appendix C provides the construction of n-variate distribution functions which satisfy conditions (1) and (2) of Theorem 4, it follows from the arguments given above that such a pair of n-variate distribution functions constitute an equilibrium. The proof of uniqueness of the univariate marginal distributions is given in Appendix A.

4 Conclusion

Kvasov (2007) introduces a non-constant-sum version of the Colonel Blotto game which relaxes the "use it or lose it" feature of the traditional constant-sum formulation of the game. In the case of symmetric budgets, that paper establishes that a suitable affine transformation of the constant-sum equilibrium is an equilibrium of the non-constant-sum game. In this paper we characterize all asymmetric parameter configurations of the non-constant-sum version of the Colonel Blotto game. As long as the player's budgets are not too asymmetric, a suitable affine transformation (with respect to the modified budgets) of the constant-sum asymmetric equilibrium (Roberson 2006) is an equilibrium of the non-constant-sum asymmetric game. However, once the players' budgets are sufficiently asymmetric this correspondence breaks down. In this parameter range, we construct entirely new equilibrium joint distributions.

Appendix A

This appendix characterizes the sets of equilibrium univariate marginal distributions in Theorems 1, 2, and 4. Given that the non-constant-sum Colonel Blotto game is a set of independent and simultaneous all-pay auctions with (symmetric and asymmetric) budget constraints, the characterization of the equilibrium univariate marginal distributions follows along the line of argument for the characterization of the all-pay auction by Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996). Roberson (2006) establishes the existence of feasible n-variate distribution functions for Theorems 1 and 2. The existence of such n-variate distribution functions for Theorem 4 is given in Appendix C.

In the discussion that follows we will focus on Theorem 1. The proofs for Theorems 2 and 4 follow directly. Let $\overline{s}_i^j$ and $\underline{s}_i^j$ denote the upper and lower bounds of player i's distribution of resources for all-pay auction j.

Recall that in Theorem 1 the corresponding parameter space is $(2/n) \leq (\overline{X}_A/\overline{X}_B) \leq 1$. It will also be convenient to note that, for a given P_{-i}, with the set of univariate marginal distribution functions $\{F_{-i}^j\}_{j=1}^n$, the Lagrangian of each player i's optimization problem[6] can be written as

$$\max_{\{F_i^j\}_{j=1}^n} (1+\lambda_i) \sum_{j=1}^{n} \left[\int_0^\infty \left[\frac{v}{(1+\lambda_i)} F_{-i}^j(x) - x \right] dF_i^j \right] + \lambda_i X_i \qquad (11)$$

where the set of univariate marginal distribution functions $\{F_i^j\}_{j=1}^n$ satisfy the constraint that there exists a mapping of the set of univariate marginal distributions into a joint distribution (an n-copula), C, such that the support of the n-variate distribution $C(F_i^1(x^1), \ldots, F_i^n(x^n))$ is contained in $\mathfrak{B}_i$.

The first two lemmas follow along the lines of the proofs for the symmetric case given in Kvasov (2007).[7]

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[6] This formulation assumes that for all battlefields the players' univariate marginal distributions do not place an atom on the same value. However, it is straightforward to incorporate the tie-breaking rule into the Lagrangian of each player's optimization problem.

[7] While the characterization of the equilibrium univariate marginal distributions for the constant-sum and non-constant-sum versions of the game follow along similar lines, there are important distinctions. In both cases, Lemmas 1-4 are established using feasible points in the support. In the non-constant-sum game Kvasov (2007) uses a sepa-

Lemma 1 *In any equilibrium* $\{F_i^j, F_{-i}^j\}_{j\in\{1,\ldots,n\}}$*, no* F_i^j *can place an atom in the half open interval* $(0,\bar{s}^j]$.

Lemma 2 *For each* $j \in \{1,\ldots,n\}$ *and for each* $i \in \{A,B\}$, $\frac{v}{1+\lambda_i}F_{-i}^j(x) - x$ *is constant* $\forall$ $x \in (0,\bar{s}^j]$.

The next two lemmas follow along the lines of the proofs in Baye, Kovenock, and de Vries (1996).

Lemma 3 *For each* $j \in \{1,\ldots,n\}$, $\bar{s}_{-i}^j = \bar{s}_i^j = \bar{s}^j$.

Lemma 4 $\forall$ $j \in \{1,\ldots,n\}$, $F_B^j(0) = 0$ *and, thus,* $\frac{v}{1+\lambda_A}F_B^j(x) - x = 0$ $\forall$ $x \in [0,\bar{s}^j]$.

The following lemma characterizes the relationship between λ_A and λ_B. Let $\bar{X}_i$ denote player i's expected expenditure, that is

$$\bar{X}_i = \sum_{j=1}^{n} \int_0^{\bar{s}^j} x dF_i^j(x). \tag{12}$$

Lemma 5 *In equilibrium* $(1+\lambda_A) = (1+\lambda_B)\frac{\bar{X}_B}{\bar{X}_A}$.

Proof From Lemma 2, it follows that $dF_A^j(x) = \frac{(1+\lambda_B)}{v}dx$ and $dF_B^j(x) = \frac{(1+\lambda_A)}{v}dx$ for all $x \in [0,\bar{s}^j]$. Substituting these expressions into equation (12), we have $(1+\lambda_A) = \bar{X}_B\frac{2v}{\sum_j(\bar{s}^j)^2}$ and $(1+\lambda_B) = \bar{X}_A\frac{2v}{\sum_j(\bar{s}^j)^2}$. The result follows directly. □

The following lemma establishes the value of $\bar{s}^j$.

Lemma 6 $\bar{s}^j = \frac{v}{1+\lambda_A}$.

Proof From Lemma 2, we know that for each player i and any battlefield j, $\frac{v}{1+\lambda_i}F_{-i}^j(x) - x$ is constant $\forall$ $x \in \left(0,\bar{s}^j\right]$. It then follows that player i would never use a strategy that provides offers in $(\frac{v}{1+\lambda_i},\infty)$ since an offer of zero strictly dominates such a strategy. It follows from Lemma 4 that $\frac{v}{1+\lambda_A} \leq \frac{v}{1+\lambda_B}$. Thus $\bar{s}^j \leq \frac{v}{1+\lambda_A}$ and $\forall$ $x \in (0,\bar{s}^j]$

$$\frac{v}{1+\lambda_i}F_{-i}^j(x) - x \geq \frac{v}{1+\lambda_i} - \bar{s}^j.$$

rating hyperplane argument to prove that each of the univariate marginal distributions is strictly increasing and continuous on its support. Conversely, in the constant-sum game Roberson (2006) relies on properties of two-player constant-sum games (namely, interchangeability of equilibrium strategies and uniqueness of equilibrium payoffs) to establish these properties of the univariate marginal distributions.

By way of contradiction, assume that $\bar{s^j} < \frac{v}{1+\lambda_A}$. By allocating a level of force to battlefield j that is greater than $\bar{s^j}$ by an arbitrarily small amount, player A can earn arbitrarily close to $\frac{v}{1+\lambda_A} - \bar{s^j} > 0$ on battlefield j, which contradicts Lemma 4. □

The following lemma establishes that there exists a unique pair λ_A, λ_B that satisfies the budget constraint.

Lemma 7 *There exists a unique value for λ_A, and thus for λ_B. $\lambda_A = \frac{nv}{2\bar{X}_B} - 1$ and thus $\lambda_B = \frac{nv\bar{X}_A}{2\bar{X}_B^2} - 1$.*

Proof The expected expenditure determines the unique pair λ_A, λ_B. Thus, λ_A solves

$$\frac{n\left(1+\lambda_A\right)}{v}\int_0^{\frac{v}{1+\lambda_A}} x\lambda_A dx = \bar{X}_B.$$

Solving for λ_A we have that

$$\lambda_A = \frac{nv}{2\bar{X}_B} - 1. \tag{13}$$

It follows directly from Lemma 5 that

$$\lambda_B = \frac{nv\bar{X}_A}{2\bar{X}_B^2} - 1. \tag{14}$$

To complete the proof of Lemma 7, recall the three possible cases: (a) neither player uses all of her available resources, (b) only (the weaker) player A uses all of her available resources, and (c) both players A and B use all of their available resources.

In case (a) $\lambda_A = \lambda_B = 0$. From (13) and (14) we have that $\bar{X}_B = \frac{nv}{2}$ and $\bar{X}_B = \sqrt{\frac{nv\bar{X}_A}{2}}$. Thus, $X_B \geq \frac{nv}{2}$ and $X_A \geq \frac{nv}{2}$. In case (b) $\lambda_A > 0$ and $\lambda_B = 0$. From (13) and (14) we have that $\bar{X}_B < \frac{nv}{2}$ and $\bar{X}_B = \sqrt{\frac{nv\bar{X}_A}{2}}$. Thus, $X_B \geq \sqrt{\frac{nvX_A}{2}}$ and $X_A < \frac{nv}{2}$. In case (c) $\lambda_A > 0$ and $\lambda_B > 0$. From (13) and (14) we have that $\bar{X}_B < \frac{nv}{2}$ and $\bar{X}_B < \sqrt{\frac{nv\bar{X}_A}{2}}$. Thus, $X_B < \sqrt{\frac{nvX_A}{2}}$ and $X_A < \frac{nv}{2}$.

To summarize $\bar{X}_B = \min\{X_B, \sqrt{\frac{nv\bar{X}_A}{2}}\}$ and $\bar{X}_A = \min\{X_A, \frac{nv}{2}\}$. Thus, for any pair X_A, X_B there exists a unique pair $\bar{X}_A$, $\bar{X}_B$, and a unique pair λ_A, λ_B. □

This completes the characterization of the sets of equilibrium univariate marginal distributions in Theorem 1. The proofs for Theorems 2 and 4 follow along similar lines.

Appendix B

The proof of Theorem 3, stated below, establishes the existence of an equilibrium in the game $NCB\{X_A, X_B, n\}$ for X_A, X_B, and $n \geq 3$ such that $\frac{\bar{X}_A}{X_B} < \frac{2}{n}$ and $\min\{nX_A, (n-2)X_A + \frac{2v}{n}\} > X_B > (n-1)X_A$. The proof of uniqueness of the equilibrium payoffs follows directly. In the discussion that follows, recall that $k = \left\lceil \frac{X_A}{X_B - X_A(n-1)} \right\rceil$, and thus, $2 \leq k < \infty$.

First, the strategies in the statement of Theorem 5 are feasible since for player A

$$(k-1-i)\frac{X_A}{k-1} + i\frac{X_A}{k-1} = X_A,$$

and for player B

$$X_A(n-2) + X_A - i\frac{nX_A - X_B}{k-1} + X_A - (k-1-i)\frac{nX_A - X_B}{k-1} = X_B$$

for all $i = 0, \ldots, k-1$.

Second, each player is indifferent between each point in the support of their strategy. For this equilibrium the univariate marginal distributions for player A and $\forall\ j \in \{1, \ldots, n\}$ are

$$F_A^j(x) = \begin{cases} \frac{n-2}{n} + \frac{\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k} + \frac{x}{v} & \text{if } x \in \left[0, \frac{X_A}{k-1}\right) \\ \frac{n-2}{n} + \frac{2\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k} + \frac{x}{v} & \text{if } x \in \left[\frac{X_A}{k-1}, \frac{2X_A}{k-1}\right) \\ \vdots & \vdots \\ \frac{n-2}{n} + \frac{(i+1)\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k} + \frac{x}{v} & \text{if } x \in \left[\frac{iX_A}{k-1}, \frac{(i+1)X_A}{k-1}\right) \\ \vdots & \vdots \\ \frac{n-2}{n} + \frac{(k-1)\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k} + \frac{x}{v} & \text{if } x \in \left[\frac{(k-2)X_A}{k-1}, X_A\right) \\ 1 & \text{if } x \geq X_A \end{cases}.$$

Similarly for player B and $\forall\ j \in \{1,\dots,n\}$ we have

$$F_B^j(x) = \begin{cases} \frac{x}{v} & \text{if } x \in [0, X_B - X_A(n-1)) \\ \frac{\left(\frac{2}{n} - \frac{X_B - X_A(n-2)}{v}\right)}{k} + \frac{x}{v} & \text{if } x \in \left[X_B - X_A(n-1), X_A - \frac{(k-2)(nX_A - X_B)}{k-1}\right) \\ \vdots & \vdots \\ \frac{(i+1)\left(\frac{2}{n} - \frac{X_B - X_A(n-2)}{v}\right)}{k} + \frac{x}{v} & \text{if } x \in \left[X_A - \frac{(k-1-i)(nX_A - X_B)}{k-1}, X_A - \frac{(k-2-i)(nX_A - X_B)}{k-1}\right) \\ \vdots & \vdots \\ \frac{(k-1)\left(\frac{2}{n} - \frac{X_B - X_A(n-2)}{v}\right)}{k} + \frac{x}{v} & \text{if } x \in \left[X_A - \frac{nX_A - X_B}{k-1}, X_A\right) \\ 1 & \text{if } x \geq X_A \end{cases}.$$

We begin with player A's expected payoff for each of her k mass points, and then examine the remaining uniform randomization. Note that for $i = 1,\dots,k-1$,[8]

$$X_A - (k-i)\frac{nX_A - X_B}{k-1} < i\frac{X_A}{k-1} \leq X_A - (k-1-i)\frac{nX_A - X_B}{k-1}.$$

Thus, given that player B is following the equilibrium strategy, player A's allocation of $i\frac{X_A}{k-1}$ to an all-pay auction yields the expected payoff

$$\frac{vi\left(\frac{2}{n} - \frac{X_B - X_A(n-2)}{v}\right)}{k}$$

for each $i = 0,\dots,k-1$. Similarly, player A's remaining resources $(k-1-i)\frac{X_A}{k-1}$ have expected payoff of

$$\frac{v(k-1-i)\left(\frac{2}{n} - \frac{X_B - X_A(n-2)}{v}\right)}{k}.$$

Thus, for each $i = 0,\dots,k-1$ player A's allocation of $((k-1-i)\frac{X_A}{k-1}, i\frac{X_A}{k-1})$ has an expected payoff of

$$\frac{v(k-1)\left(\frac{2}{n} - \frac{X_B - X_A(n-2)}{v}\right)}{k}.$$

Lastly, we consider player A's expected payoff from the uniform randomization between the mass points. Given that player B is following the equilibrium strategy, the payoff to player A for any allocation in which no all-pay auction is allocated more than $X_B - X_A(n-1)$ is zero. Similarly, if, for any $0 < \varepsilon \leq \frac{nX_A - X_B}{k-1}$ and for some $i =$

[8] For the remaining case that $i = 0$, $0 < X_B - X_A(n-1)$.

$1,\ldots,k-2$,[9] player A allocates $X_A-(k-1-i)\frac{nX_A-X_B}{k-1}+\varepsilon$ to an all-pay auction the expected payoff for that all-pay auction is $v(i+1)\left(\frac{2}{n}-\frac{X_B-X_A(n-2)}{v}\right)/k$. Player A's remaining resources are $(k-1-i)\frac{nX_A-X_B}{k-1}-\varepsilon$ and

$$(k-1-i)\,\frac{nX_A-X_B}{k-1}-\varepsilon\leq X_A-(i+1)\,\frac{nX_A-X_B}{k-1}$$

since, from the definition of k, $nX_A-X_B\leq X_A\frac{k-1}{k}$. If player A allocates all of her remaining resources to a single all-pay auction the maximum expected payoff for that all-pay auction is $v(k-i-2)\left(\frac{2}{n}-\frac{X_B-X_A(n-2)}{v}\right)/k$. Thus, for player A any feasible allocation of force in which only 1 or 2 all-pay auctions receive a strictly positive level of force has a maximum expected payoff of $v(k-1)\left(\frac{2}{n}-\frac{X_B-X_A(n-2)}{v}\right)/k$. In addition, since the step size between each mass point in player B's equilibrium strategy is $\frac{nX_A-X_B}{k-1}$, player B's minimal mass point is at $X_B-X_A(n-1)\geq\frac{nX_A-X_B}{k-1}$, and each mass point has the same weight, player A cannot achieve a higher expected payoff from dividing these remaining resources among more than one all-pay auction. Thus, given that player B is following the equilibrium strategy, the maximum expected payoff to player A for an arbitrary strategy $\mathbf{x}\in\mathfrak{B}_A$ is

$$\sum_{j=1}^{n}\left[vF_B^j\left(x_j\right)-x_j\right]\leq\frac{v\,(k-1)\left(\frac{2}{n}-\frac{X_B-X_A(n-2)}{v}\right)}{k}.$$

The argument for player B is symmetric.

This completes the proof of Theorem 3.

Appendix C

Subject to the constraint that there exist sufficient n-variate distribution functions, Theorems 1, 2, and 4 characterize the unique sets of equilibrium univariate marginal distribution functions for their respective parameter ranges.

For Theorems 1 and 2 Roberson (2006) demonstrates the existence of such n-variate distribution functions. This Appendix establishes the existence of sufficient n-variate distributions for the Theorem 4 parameter range.

[9] For the remaining case that $i=k-1$, player A's payoff from allocating all X_A to a given all-pay auction is the same as if player A allocates $X_A-\frac{nX_A-X_B}{k-1}+\varepsilon$ to the all-pay auction. This follows from the tie-breaking rule and the fact that in this case player A's remaining resources are $\frac{nX_A-X_B}{k-1}-\varepsilon$, and $\frac{nX_A-X_B}{k-1}-\varepsilon<X_B-X_A(n-1)$, for all admissible k and $\varepsilon>0$, so that the payoff from player A's remaining resources is 0.

Theorem 5 *For each unique set of equilibrium univariate marginal distribution functions, $\{F_i^j\}_{j=1}^n$, characterized in Theorem 4, there exists an n-copula, C, such that the support of the n-variate distribution function $C(F_i^1(x^1),\ldots,F_i^n(x^n))$ is contained in $\mathfrak{B}_i$.*

We begin with the proof for player A. The construction of a sufficient n-variate distribution function for player A and $X_A \geq \frac{v}{n}$ is outlined as follows (recall that in the Theorem 4 parameter region $X_A < \frac{2v}{n}$). The remaining case that $X_A < \frac{v}{n}$ is addressed directly following this case.

1. Player A selects $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability, and provides zero resources to those all-pay auctions.
2. On the remaining 2 all-pay auctions, player A randomizes uniformly on three line segments: (i) $\{(x_1,x_2)|\ x_1+x_2=2X_A-\frac{2v}{n}\}$, (ii) $\{(x_1,x_2)|\ x_1=0 \text{ and } 2X_A-\frac{2v}{n}\leq x_2\leq X_A\}$, and (iii) $\{(x_1,x_2)|\ x_2=0 \text{ and } 2X_A-\frac{2v}{n}\leq x_1\leq X_A\}$. This support is shown in Panel (ii) of Figure 2, and this randomization is discussed in greater detail directly following this outline.
3. There are ${}_nC_2$ ways of dividing the n all-pay auctions into disjoint subsets such that $n-2$ all-pay auctions receive zero resources with probability 1 and 2 all-pay auctions involve randomizations of resources as in point 2. The n-variate distribution function formed by placing probability $[{}_nC_2]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $(1-\frac{X_A}{v})$ at 0 and randomize uniformly on $(0,X_A]$ with the remaining mass.

The pivotal step in this construction is point 2. Let x_i denote the allocation of resources to all-pay auction $i\in\{1,2\}$. Consider the support of a bivariate distribution function, F, for x_1 and x_2 which uniformly places mass $1-\frac{nX_A}{2v}$ on each of the two following line segments

$$\{(x_1,x_2)|\ x_1=0 \text{ and } 2X_A-\frac{2v}{n}\leq x_2\leq X_A\}$$
$$\{(x_1,x_2)|\ x_2=0 \text{ and } 2X_A-\frac{2v}{n}\leq x_1\leq X_A\}$$

and uniformly places the remaining mass, $\frac{nX_A}{v}-1$, on the line segment

$$\{(x_1,x_2)|\ x_1+x_2=2X_A-\frac{2v}{n}\}$$

This support is shown in Panel (ii) of Figure 2.

[Insert Figure 2 here]

In the expression for this bivariate distribution function we will use the following notation.

R1: $\left\{(x_1,x_2) \in [0, 2X_A - \frac{2v}{n}]^2\right\}$
R2: $\left\{(x_1,x_2) \in [2X_A - \frac{2v}{n}, X_A] \times [0, 2X_A - \frac{2v}{n}]\right\}$
R3: $\left\{(x_1,x_2) \in [0, 2X_A - \frac{2v}{n}] \times [2X_A - \frac{2v}{n}, X_A]\right\}$
R4: $\left\{(x_1,x_2) \in (2X_A - \frac{2v}{n}, X_A]^2\right\}$

The bivariate distribution function for x_1, x_2 is given by

$$F(x_1,x_2) = \begin{cases} \left(\frac{n}{2v}\right) \max\left\{x_1 + x_2 - 2X_A + \frac{2}{vn}, 0\right\} & \text{if } (x_1,x_2) \in \text{R1} \\ \left(1 - \frac{nX_A}{v}\right) + \frac{nx_1}{2v} + \frac{nx_2}{2v} & \text{if } (x_1,x_2) \in \text{R2} \cup \text{R3} \cup \text{R4} \end{cases}$$

The univariate marginal distributions are given by $F(x_1, X_A) = (1 - \frac{nX_A}{2v}) + \frac{nx_1}{2v}$ and $F(X_A, x_2) = (1 - \frac{nX_A}{2v}) + \frac{nx_2}{2v}$. To see that F provides the necessary univariate marginal distributions, observe that given the randomization outlined above player A allocates zero resources to each all-pay auction j with probability $\frac{n-2}{n} + \frac{2}{n}(1 - \frac{nX_A}{2v}) = (1 - \frac{X_A}{v})$ and randomizes uniformly over the interval $(0, X_A]$ with the remaining mass.

If $X_A < \frac{v}{n}$, then player A allocates zero resources to $n-1$ of the all-pay auctions and provides a random level of resources in the one remaining all-pay auction. In this one remaining all-pay auction player A has a mass point of $(1 - \frac{nX_A}{v})$ at 0 and randomizes uniformly over the interval $[0, X_A]$ with the remaining mass.

The proof for player B is similar. The construction of a sufficient n-variate distribution function for player B and $X_A \geq \frac{v}{n}$ is outlined as follows. In the Theorem 4 parameter region $X_B \geq \min\{nX_A, (n-2)X_A + (2v/n)\}$. If $X_A \geq \frac{v}{n}$ then $X_B \geq (n-2)X_A + (2v/n)$. The remaining case that $X_A < \frac{v}{n}$ and $X_B \geq nX_A$ is addressed directly following this case.

1. Player B selects $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability, and allocates X_A to each of those all-pay auctions.
2. On the remaining 2 all-pay auctions, player B randomizes uniformly on three line segments: (i) $\{(x_1,x_2) | \, x_1 + x_2 = \frac{2v}{n}\}$, (ii) $\{(x_1,x_2) | \, x_1 = X_A \text{ and } 0 \leq x_2 \leq \frac{2v}{n} - X_A\}$, and (iii) $\{(x_1,x_2) | \, x_2 = X_A \text{ and } 0 \leq x_1 \leq \frac{2v}{n} - X_A\}$. This support is shown in Panel (i) of Figure 2, and this randomization is discussed in greater detail directly following this outline.
3. There are ${}_nC_2$ ways of dividing the n all-pay auctions into disjoint subsets such that $n-2$ all-pay auctions receive X_A with probability 1 and 2 all-pay auctions involve randomizations of force as in point 2. The n-variate distribution function formed by placing probability $[{}_nC_2]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $(1 - \frac{X_A}{v})$ at X_A and randomize uniformly on $[0, X_A)$ with the remaining mass.

The pivotal step in this construction is again point 2. Let x_i denote the allocation to all-pay auction $i \in \{1,2\}$. Consider the support of a bivariate distribution function, F, for x_1 and x_2 which uniformly places mass $1-\frac{nX_A}{2v}$ on each of the two following line segments

$\{(x_1,x_2)|\ x_1 = X_A \text{ and } 0 \leq x_2 \leq \frac{2v}{n} - X_A\}$
$\{(x_1,x_2)|\ x_2 = X_A \text{ and } 0 \leq x_1 \leq \frac{2v}{n} - X_A\}$

and uniformly places the remaining mass, $\frac{nX_A}{v}-1$, on the line segment

$\{(x_1,x_2)|\ x_1 + x_2 = \frac{2v}{n}\}$

This support is shown in Panel (i) of Figure 2.

The bivariate distribution function for x_1, x_2 is given by

$$F(x_1,x_2) = \begin{cases} \left(\frac{n}{2v}\right)\max\left\{x_1 + x_2 - \frac{2}{vn}, 0\right\} & \text{if } (x_1,x_2) \in [0,X_A)^2 \\ \frac{nx_1}{2v} & \text{if } x_2 = X_A,\ x_1 \in [0,X_A) \\ \frac{nx_2}{2v} & \text{if } x_1 = X_A,\ x_2 \in [0,X_A) \\ 1 & \text{if } x_1, x_2 \geq X_A \end{cases}$$

Following from the arguments given above for player A, it follows that F provides the necessary univariate marginal distributions for all-pay auctions 1 and 2.

If $X_A < \frac{v}{n}$ and $X_B \geq nX_A$, then player B allocates X_A to $n-1$ of the all-pay auctions and provides a random level of resources in the one remaining all-pay auction. In this one remaining all-pay auction player A has a mass point of $(1-\frac{nX_A}{v})$ at X_A and randomizes uniformly over the interval $[0,X_A)$ with the remaining mass.

This completes the proof of the existence of sufficient n-variate distributions for the Theorem 4 parameter range.

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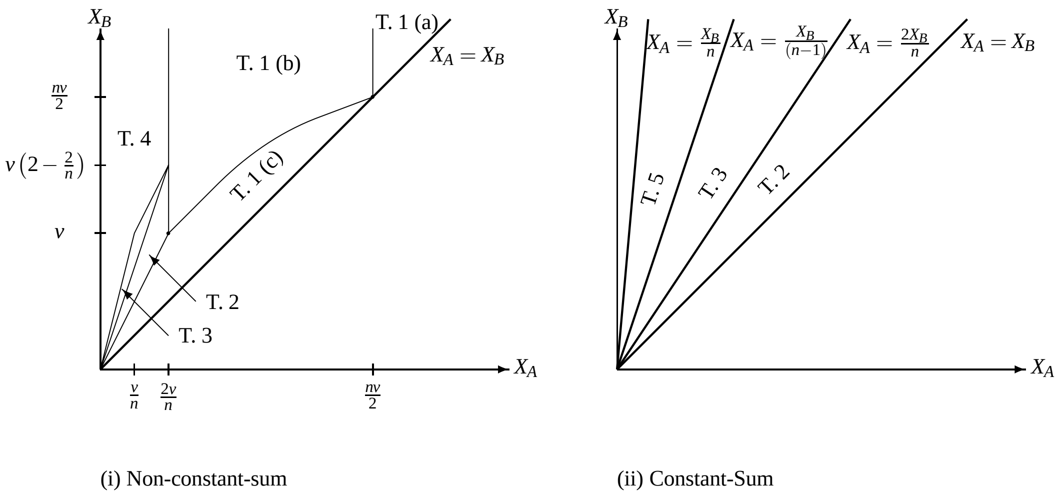

Fig. 1 Resource Level Configurations

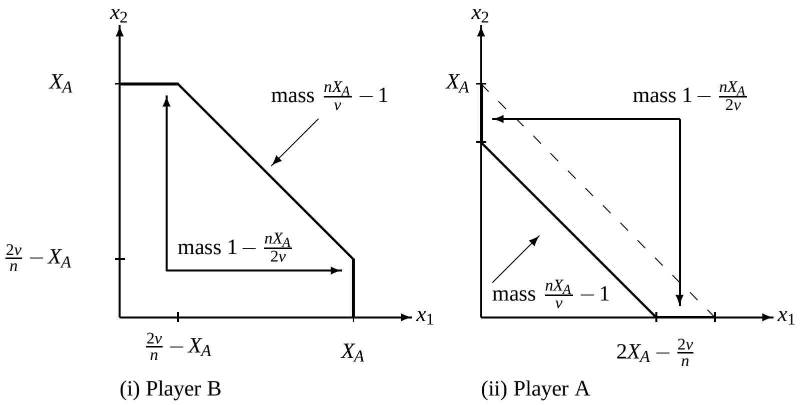

Fig. 2 Support of players' bivariate distributions $((\bar{X}_A/\bar{X}_B) < (2/n)$, $X_A > (v/n)$ and $X_B > (n-2)X_A + (2v/n))$