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Contests with Rank-Order Spillovers

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Abstract

This paper presents a unified framework for analyzing equilibrium in simultaneous move, two-player, rank-order contests with complete information, in which each player's strategy generates direct or indirect affine "spillover" effects that depend on the rank-order of her decision variable. These effects arise in natural interpretations of a number of important economic environments, as well as in classic contests adapted to recent experimental and behavioral models where individuals exhibit inequality aversion or regret. We provide the closed-form solution for the symmetric equilibrium to this class of games, and show how it may be used to directly solve for equilibrium behavior in auctions, pricing games, tournaments, R&D races, litigation, and a host of other games.

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1 Introduction

This paper presents a unified framework for analyzing equilibrium in simultaneous-move, two-player, rank-order contests with complete information, in which each player's strategy generates direct or indirect affine "spillover" effects that depend on the rank-order of her decision variable. These effects arise in natural interpretations of a number of important economic environments, including contests adapted to recent experimental and behavioral models where individuals exhibit inequality aversion or regret. We provide the closed-form solution for the symmetric equilibrium to this class of games, and show how it may be used to directly solve for equilibrium behavior in auctions, pricing games, tournaments, R&D races, litigation, and a host of other games.

Rank-order contests are ubiquitous in life. These take the form of environments in which players choose nonnegative bids (which may be interpreted as a proposed payment, effort, or the commitment of other scarce resources that are non-refundable) whose rank-order discontinuously influences the probability of winning some prize. Classic examples include homogeneous good Bertrand competition (Bertrand, 1883), in which the lowest price firm "wins" the profit from selling to demand at that price, and both first and second-price auctions (Vickrey, 1961), where the player who submits the highest bid wins the auctioned object and pays either his own bid (in the first-price auction) or the bid of the second-highest bidder (in the second-price auction).

Other forms of rank-order contests involve both winners and losers alike forfeiting payments. In a first-price all-pay auction, for instance, each player submits a non-refundable bid and only the higher bidder receives a prize. The war-of-attrition (Maynard Smith, 1974) is a second-price all-pay auction: the high bidder wins the prize and pays the amount bid by the second-highest bidder. These forms of competition have been widely used to model activities as diverse as patent and R&D races, lobbying and rent-seeking activities, litigation, advertising and political campaigns, tournaments as incentive devices in labor markets, competition for college admissions, sports competitions, urban architecture, and territorial contests among organisms. [1]

The main motivation of this article is that in many rank-order contests *spillovers* are important. That is, one player's decision variable often affects the other player's payoff, and the nature of this effect may depend on the rank-order of the players' choices.

This is perhaps most obvious in second-price auctions where the high bidder pays the second highest bidder's bid. However, spillovers arise in other important economic contexts. For instance, an extensive literature, starting with D'Aspremont and

[1] Applications in these areas include work by Dasgupta (1986), Kaplan, Luski and Wettstein (2003), Hillman and Riley (1989), Baye, Kovenock and de Vries (1993), Che and Gale (1998), Baye, Kovenock and de Vries (2005), Sahuguet and Persico (2006), Konrad (2004), Fu (2006), Groh, Moldovanu, Sela and Sunde (forthcoming), Helsley and Strange (2008), and Kura (1999).

Jacquemin (1988), has examined the effects of positive spillovers in R&D competition. These arise when one player's R&D effort provides information that benefits its rival. Although D'Aspremont and Jacquemin (1988) does not involve rank-order effects, a growing literature, starting from an original observation by Dasgupta (1986), models the R&D process or as a rank-order tournament (see also Che and Gale (2003) and Zhou (2006)). The results examined in this article apply to the positive informational spillovers arising in this context.

Rank-order dependent spillovers also arise in models of litigation. Baye, Kovenock and de Vries (2005) examine equilibrium in a litigation game in which legal expenditures increase the quality of the case presented and the best case wins. This turns the litigation process into a rank order contest in which the behavior of various legal systems, such as the American, British, Continental and "Quayle" systems, may be examined. Although the American system, in which litigants pay their own legal costs, involve no spillovers, other fee-shifting rules, such as the British and Continental rules, which require that losers compensate winners for a portion of their legal costs, and the Quayle system, in which the loser reimburses the winner up to the amount actually spent by the loser, involve spillovers. Under the British and Continental rules there is an indirect spillover effect of the winner's expenditure on the loser that is negative. In the continuation we call this a *second-order negative spillover* effect. In the case of the Quayle system, there is an indirect spillover of the loser's expenditure on the winner that is positive. We call this a *first-order positive spillover* effect.

Our taxonomy of spillover effects may also be used to construct and analyze other variants and extensions of the auction literature noted above. For instance, the classic partnership dissolution problem may be viewed as the auction of a business in which two partners simultaneously submit bids and the partner with the higher bid pays his bid to the partner with the lower bid in return for ownership of the business. In this case, the payment of the winning partner is a *second-order positive spillover* effect on the loser. Similarly, both the first-price and second-price all-pay auctions, often applied in the context of biological contests, may be extended to include environments in which effort expended imposes both a rank-order contingent direct effect on the player expending the effort and a rank order spillover effect on the player's rival. For instance, if two organisms are engaged in a territorial fight, the effort of the winner may exact both a cost to the winner (a *first-order negative direct* effect) and a cost to the loser (a *second-order negative spillover* effect). The loser's effort may have a *second-order negative direct* effect on the loser's payoff and a *first-order negative spillover* effect on the winner.

An important class of economic environments in which rank-order dependent spillovers arise naturally is the analysis of auctions adapted to recent experimental and behavioral models of individual choice. In Section 4 we apply our characterization to examine three such models: (i) a job tournament in which individuals exhibit Fehr-Schmidt (1999) inequality aversion, (ii) an extension of the the Filiz-Ozbay and

Ozbay (2007) models of winner and loser anticipated regret in first-price auctions to cover combined winner-loser regret, and (iii) an all-pay auction in which players maximize relative fitness according to the finite agent Evolutionary Stable Strategies (ESS) equilibrium of Schaffer (1988)[2].

Finally, many pricing games have rank-order dependent spillovers. A variant of the classic Bertrand model, due to Varian (1980), has two sellers simultaneously setting a price and selling to three inelastic segments of demand with common choke price r. One of these inelastic segments consists of price sensitive consumers who are aware of both prices in the market and who purchase from the lower-price seller, while the other two segments are attached to different firms and are each aware of only the price of that firm to which they are attached (as long as that price is at or below the choke price). Baye, Kovenock and de Vries (1992) have shown that this game has a structure similar to that of a first-price all-pay auction in which the bid is the difference between the choke price r and the price set by the player, p_i. In this context, the bid corresponds to the opportunity cost of the lost revenue from the seller's own uninformed segment that results from reducing price in order to attempt to capture the "prize" consisting of the demand of the informed price-sensitive consumer segment.

Spillovers arise naturally in the context of the Varian model when one examines popular price matching policies (see Lin (1988), Png and Hirshleifer (1987), Baye and Kovenock (1994)). If a high price seller i institutes a price matching policy it will sell at its own price p_i to consumers informed only of that price, but sell at its rival's price $p_j < p_i$ to a proportion $\mu \leq 1/2$ of the informed customers who are willing to bear the cost of visiting the high price seller and taking it up on its offer to match the better price. In this case the rival's low price, p_j, generates a spillover effect on the high price seller's payoff, but not vice-versa. In Section 4 we also provide an example of a "reference pricing" version of the Varian model in which a segment of "relative bargain" seekers exists, whose demand from the low-price firm depends on the ratio of the high price to the low price. In this case a rival's high price generates a spillover effect on low price seller's payoff, but not vice-versa.

All of these models have the property that they fit into a single linear parameterized class of rank-order contests whose equilibria may be characterized in a unified fashion according to whether the defining parameters lie in a small, finite number of regions of the parameter space. In Section 2 we introduce this class of models and provide a general closed-form solution for the symmetric equilibria of the class. We characterize equilibrium bidding strategies as functions of “contest parameters” that can be varied to change the “rules” of the contest. In Section 3, we show how this characterization may be used to directly solve for symmetric equilibrium behavior in the economic environments discussed. In Section 4 we conclude. The proofs are collected in the Appendix.

[2] See also Hehenkamp, Leininger, and Possajennikov (2004) who were the first to apply this solution concept in the contest literature in their analysis of the ESS of the Tullock contest.

2 Model and Results

We study the general solution to the game with the following payoffs

$$\begin{cases} W(x_i, x_j) = v - \beta x_i - \delta x_j & \text{if } x_i > x_j \\ L(x_i, x_j) = -\gamma - \alpha x_i - \theta x_j & \text{if } x_i < x_j \\ T(x_i, x_j) = \frac{1}{2} W(x_i, x_j) + \frac{1}{2} L(x_i, x_j) & \text{if } x_i = x_j \end{cases} \quad (1)$$

where each player $i, i = 1, 2$, selects a bid x_i from the strategy space $A = [0, \infty)$. We assume that $v + \gamma \geq 0$ and $v \geq 0$. Otherwise, one could simply redefine winning as losing and vice versa. Note that this implies $W(0,0) \geq L(0,0)$, with strict inequality if $v + \gamma > 0$. In order to avoid a game of "pick the highest number" we assume that at least one of the parameters β, δ, α, or θ has to be nonzero.

The δ and θ parameters capture the externalities (negative or positive) that contestants may inflict on each other. We use the terms "first-order positive (negative) *spillover* effects" when $\delta < (>) \, 0$, and "second-order positive (negative) spillover effects" when $\theta < (>) \, 0$. This captures the fact that when player i's bid x_i is the higher bid, or first, in the rank-order, the *spillover* effect of player j's bid, $j \neq i$, on player $i's$ payoff is linear with coefficient δ. If $\delta > 0$, this effect is negative and if $\delta < 0$ this effect is positive. Likewise, when player i's bid x_i is the lower bid, or second, in the rank-order, the spillover effect of player j's bid, $j \neq i$, is linear with coefficient θ. If $\theta > 0$, this effect is negative and if $\theta < 0$ this effect is positive. For similar reasons, we refer to β and α as the first- and second-order *direct* effects. If player i's bid x_i is the higher bid, or first, in the rank-order, the *direct* effect of player i's bid on player $i's$ payoff is linear with coefficient $-\beta$. If $\beta > 0$, the first order direct effect of an increase in player i's bid is negative and is positve if $\beta < 0$. Similar interpretations apply to the second-order direct effect α.

The game with payoffs (1) can be analyzed within the general existence framework offered by Dasgupta and Maskin (1986), except that we allow for unbounded strategies. The framework also fits within the differential equation solution approach of Lizzeri and Perisco (2000) to existence and uniqueness of bidding strategies in auctions. Lizzeri and Perisco (2000) consider general payoff functions $W(x_i, x_j)$ and $L(x_i, x_j)$, in a setting of incomplete information with interdependent valuations. This article addresses the linear specification in the complete information setting, provides the specific solutions and includes effects not covered by the Lizzeri-Persico analysis.[3]

2.1 Pure Strategy Nash Equilibria

We provide the conditions under which there exists a symmetric pure strategy equilibrium, (x, x), such that each player earns the equilibrium payoff $U^* = T(x, x) =$

[3]In particular, complete information analogues of the Lizzeri-Persico axioms would require $\alpha \geq 0$ (A5, A7), $\beta > 0$ (A7), $\delta \geq 0$ (A7), and $\theta = 0$ (A7).

$W(x,x) = L(x,x)$. In the sequal we use the following abbreviations: $V \equiv v + \gamma$ and $\eta \equiv \alpha + \theta - \beta - \delta$.

Proposition 1 *Suppose $V \geq 0$, $A = [0, \infty)$ and W, L, and T are in the general linear class as described in (1). Then x is a symmetric pure strategy Nash equilibrium if and only if the following three conditions hold:*
(a) $\beta \geq 0$
(b) $x\alpha \leq 0$
(c) $V + \eta x = 0$.

Proof. See Appendix. ■

The following proposition follows immediately from the above proposition:

Proposition 2 *Consider the cases $V = 0$ and $V > 0$ separately.*
(a) Suppose $V = 0$. Then (1) has a symmetric pure strategy Nash equilibrium if and only if $\beta \geq 0$. Furthermore, when $\alpha > 0$ or $\eta \neq 0$, this equilibrium is unique within the class of symmetric equilibria and is given by $x = 0$. If $\alpha \leq 0$ and $\eta = 0$, any $x \in [0, \infty)$ is a symmetric pure strategy Nash equilibrium.
(b) Suppose $V > 0$. Then (1) has a symmetric pure strategy Nash equilibrium if and only if $\beta \geq 0, \alpha \leq 0$ and $\eta < 0$. Furthermore, this equilibrium is unique within the class of symmetric equilibria and is given by

$$x = \frac{V}{-\eta} = \frac{v + \gamma}{\beta + \delta - \alpha - \theta}. \tag{2}$$

2.2 Symmetric Mixed Strategy Equilibria

Let $F(w)$ be the cumulative distribution function of a symmetric equilibrium mixed strategy. Differentiating the incremental gain from varying one's effort and equating this to zero in equilibrium yields a differential equation that can be solved to give the symmetric equilibrium mixed strategy.

Proposition 3 *For the game with the affine payoff schedules W, L and T as defined in (1), suppose there exists a symmetric mixed strategy equilibrium with the equilibrium distribution $F(w)$. Furthermore, suppose $F(w)$ is at least once differentiable on the open subset (m, u) of the support; where $m \geq 0$ and possibly $u = \infty$. Then*

$$F(w) = \frac{\alpha}{\alpha - \beta}\left\{1 - \left[\frac{V + \eta m}{V + \eta w}\right]^{\frac{\alpha - \beta}{\eta}}\right\} + C\left[\frac{V + \eta m}{V + \eta w}\right]^{\frac{\alpha - \beta}{\eta}} \tag{3}$$

where $m \geq 0, 1 \geq C \geq 0, F(m) \geq 0, V \equiv v + \gamma \geq 0$ and $\eta \equiv \alpha + \theta - \beta - \delta$.

Proof. See Appendix. ■

Note that this only characterizes the solution by the first-order condition. Not all parameter combinations actually yield a well defined distribution function and the second-order condition has to be verified. This is done in the Appendix. Moreover, the Appendix fills in further details of the mixed strategy solutions like the lower and upper bounds. Below we summarize the viable equilibrium configurations.

2.2.1 Summary of mixed strategy equilibria

Proposition 3 gives the general solution (3) for the symmetric mixed strategy equilibrium to (1). Details of the different strategies and expected payoffs for different parameter values are provided in the Appendix. The parameter ranges yielding non-degenerate mixed strategies and the corresponding functional forms for these ranges are summarized below. The first property of the equilibria is that: *In any mixed strategy Nash equilibrium, $\beta \geq 0$ necessarily.*

There are two main configurations to be considered: $\alpha \neq 0$ and $\alpha = 0$. The case $\alpha \neq 0$ is further subdivided into the cases $V > 0$ and $V = 0$. Let m denote the lower bound of the distribution function.

Configuration $\alpha \neq 0$ and $V > 0$. The mixed strategy equilibria have no mass points and

$$F(w) = \begin{cases} \frac{\alpha}{\alpha-\beta}\left(1 - \left(\frac{V+\eta m}{V+\eta w}\right)^{\frac{\alpha-\beta}{\eta}}\right) & \text{if } \eta \neq 0; \alpha-\beta \neq 0 \\ \frac{\alpha}{\theta-\delta}\ln\left(\frac{V+(\theta-\delta)w}{V}\right) & \text{if } \eta \neq 0; \alpha-\beta = 0 \\ \frac{\alpha}{\alpha-\beta}\left(1-\exp\left(-\frac{\alpha-\beta}{V}w\right)\right) & \text{if } \eta = 0; \alpha-\beta \neq 0 \\ \frac{\alpha}{V}w & \text{if } \eta = 0; \alpha-\beta = 0 \end{cases}. \tag{4}$$

In the first configuration $\eta \neq 0$ and $\alpha \neq \beta$. Then if $\beta = 0$ and for $\alpha > 0$, η can be positive or negative and $m = 0$; but $\beta = 0$ while $\alpha < 0$, implies that $\eta < 0$ and $m > -V/\eta > 0$. If $\beta > 0$, then $\alpha > 0$ necessarily and $m = 0$, but η can be positive or negative.

In the second configuration $\eta \neq 0$ and $\alpha = \beta$. It is then necessarily the case that $\alpha > 0$ and $m = 0$, while η can be positive or negative.

In the third configuration $\eta = 0$ and $\alpha \neq \beta$. In this configuration $\beta > 0$ and $\beta = 0$ are both possible, but $\alpha > 0$ necessarily and $m = 0$.

In the fourth configuration $\eta = 0$ and $\alpha = \beta$. It is the case that $\alpha > 0$ and $m = 0$.

Configuration $\alpha \neq 0$ and $V = 0$. There only exists a mixed strategy equilibrium in case $\eta \neq 0$, $\alpha \neq \beta$ and $\beta = 0$. In this case the mixed strategy solution reads

$$F(w) = 1 - \left[\frac{m}{w}\right]^{\alpha/\eta} + C\left[\frac{m}{w}\right]^{\alpha/\eta}.$$

Furthermore, if $\alpha > 0$ then $\eta > 0$ and a mass point $C > 0$ is necessary. In the other case if $\alpha < 0$ then $\eta < 0$ and $C \geq 0$.

Configuration $\alpha = 0$ and $V > 0$. There are no mixed strategy equilibria.

Configuration $\alpha = 0$ and $V = 0$. The mixed strategy equilibria exist for two configurations of parameters:

$$F(w) = \begin{cases} Cw^{\beta/\eta} & \text{if} \quad \eta > 0; \beta > 0 \\ \textit{any} \text{ well defined cdf } F(w) & \text{if} \quad \eta = 0; \beta = 0 \end{cases}.$$

The relationship between the different solutions There is a simple way to relate the different ranges of the parameter space and the corresponding functional forms to one another. First consider the configuration $\alpha \neq 0$ *and* $V > 0$. The first solution when $\eta \neq 0$ and $\alpha \neq \beta$ follows by setting $C = 0$ in (3). As we show in the Appendix, the log-solution for the case $\eta \neq 0$ *and* $\alpha = \beta$ follows by taking appropriate limits of the first solution by letting $\alpha - \beta$ tend to zero. The third solution has $\eta = 0$ *and* $\alpha \neq \beta$, also follows by taking appropriate limits of the first solution by letting $\eta \to 0$. The last case with $\eta = 0$ *and* $\alpha = \beta$, follows from the second case as $(\theta - \delta) \to 0$; alternatively, it also follows from the third case as $(\alpha - \beta) \to 0$. For the configuration $\alpha \neq 0$ *and* $V = 0$, the solution simply follows by setting $V = 0$ in the general solution to the differential equation. The last case with $F(w) = Cw^{\beta/\eta}$ for the configuration $\alpha = 0$ *and* $V = 0$ also simply follows from the general solution by setting $\alpha = V = 0$.

2.2.2 economic interpretation of solutions

To obtain and economic interpretation of these cases, start with the first *configuration* $\alpha \neq 0$ *and* $V > 0$ in (4). There are no mass points. The last case with $\eta = 0$ and $\alpha = \beta$ is in essence the first price version of the all-pay auction, where θ and δ are nuisance parameters that cause a spillover in the expected payoff, but which do not influence the mixed strategy. If the nuisance parameters are unequal, these parameters cause an externality that does exert an influence over the mixed strategy, this gives the log-solution (still $\alpha = \beta > 0$). Recall that the fourth case is the limit of the second case as the spillovers become symmetric. So the second case can be interpreted as the all-pay auction with an asymmetric spillover. The fourth case is also the limit of the third case. If $\beta = 0$ the third case is a war of attrition; the standard war of attrition has $\alpha = \delta = 1$, $\beta = \theta = 0$. The war of attrition is also known as the second price version of the all-pay auction, and hence the fact that the fourth case is particular limit of the third case. In the third case if $\beta > 0$, the mixed strategy solution has bounded support, whereas with $\beta = 0$ the support is unbounded and has an exponential distribution.

The first case with $\eta \neq 0$ and $\alpha \neq \beta$ is in a sense the most general all pay auction when the spillovers and own effects are all asymmetric. When $\beta = 0$, there are two subcases. If $\alpha > 0$, then $m = 0$ and one obtains the sad loser auction in which only the loser pays for his bid, plus possibly some spillovers. The solution has unbounded support and the distribution is known as a Burr distribution and has a Pareto type upper tail so that not all moments are bounded. If $\alpha < 0$, necessarily $m > 0$ and the game is a version of the Baye-Morgan (1999) Bertrand price competition game. In case $\beta < 0$ there is no Nash equilibrium. But if $\beta > 0$ there exists a mixed strategy solution.

For the *configuration* $\alpha \neq 0$ *and* $V = 0$ there is only a solution in the case in which there is no direct effect of own actions when winning: $\beta = 0$. The case $V = 0$ includes the case when there is no prize for winning or losing, so that the payoffs are entirely determined by the bids and the spillovers. In this case the mixed strategy solution necessarily involves some positive mass at zero if $\alpha > 0$. Furthermore necessarily $\delta < 0$ and the size of the mass point is $\alpha / (\alpha - \delta)$. The support is unbounded. With $\alpha < 0$ the mass point is not necessary, but can exist. Since the distributions in these cases are all Pareto type, again not all moments do exist. The solutions are most directly linked to the Baye-Morgan type Bertrand price competition solution. With $V > 0$, $\alpha < 0$ and $\beta = 0$, necessarily $m > 0$ and

$$F(w) = 1 - \left(\frac{V + \eta m}{V + \eta w} \right)^{\alpha/\eta}.$$

If we take $V = 0$ (and $\alpha < 0$) in this expression, we get

$$F(w) = 1 - (m/w)^{\alpha/\eta}.$$

Interestingly, with $V > 0$ and $\alpha > 0$ and $\beta = 0$ requires $m = 0$ and the sad looser solution type emerges

$$F(w) = 1 - \left(\frac{V}{V + \eta w} \right)^{\alpha/\eta}.$$

While the special case with $V = 0$ requires a mass point at some $m > 0$ and with $\delta < 0$

$$1 - \left[\frac{m}{w}\right]^{\alpha/\eta} + \frac{\alpha}{\alpha - \delta} \left[\frac{m}{w}\right]^{\alpha/\eta} = 1 - \frac{-\delta}{\alpha - \delta} \left[\frac{m}{w}\right]^{\alpha/\eta}.$$

The last *configuration* $\alpha = 0$ *and* $V = 0$ only has a mixed strategy solution if $\beta > 0$ and $V = 0$.

$$F(w) = C w^{\beta/\eta}.$$

This is a variation on a winner pay auction with spillovers and identical prizes for the winner and loser.

3 Applications

We discuss applications to contests, auctions and pricing with externalities.

3.1 Auctions and Contests with Asymmetric Spillovers

Partnership dissolution (the self-auction) Suppose two partners want to dissolve a partnership worth v to both. The partners simultaneously place bids x_i. The high bidder wins the asset and pays his bid to the low bidder; ties are broken with the flip of a fair coin. The payoffs are given by

$$u_i(x_i, x_j) = \begin{cases} v - x_i & \text{if} \quad x_i > x_j \\ \frac{1}{2}(v - x_i) + \frac{1}{2}x_j & \text{if} \quad x_i = x_j \\ x_j & \text{if} \quad x_\iota < x_j \end{cases}$$

In this case, $V = v > 0$, $\beta = -\theta = 1, \alpha = \delta = 0$, and $\eta = -2$. Consequently, the self-auction satisfies the three conditions of Proposition 1. Applying part c of the Proposition 1 yields $V + \eta x = 0$ or $x = V/2$. Furthermore, there does not exist a symmetric mixed startegy equilibrium.

Innovation contests with informational spillovers All-pay auctions with asymmetric spillovers arise naturally in two types of economic models of contests. First consider an extension of Dasgupta's (1986) all-pay auction model of an R&D race to incorporate informational spillovers. In Dasgupta's model two players simultaneously sink resources in order to win the innovation race. The player who has the higher expenditure wins the race and both players pay their own expenditures.

Suppose, however, that each unit of expenditure generates a positive informational spillover that benefits the rival player, with the winner receiving a greater benefit per unit from the loser's expenditure than the loser receives per unit from the winner's expenditure (due to the fact that the winner may apply the spillover to the current innovation, as well as any other innovations that might arise from the advantage of winning the current innovation). In this contest, payoffs are

$$u_i(x_i, x_j) = \begin{cases} v - x_i - \delta x_j & if \quad x_i > x_j \\ -x_i - \theta x_j & if \quad x_\iota < x_j \end{cases}$$

where $V = v > 0, \alpha = \beta = 1$, and $\delta < \theta < 0$. In this case, since $V > 0$, $\alpha - \beta = 0$, and $\eta > 0$, case 2 of solution (4) applies and the symmetric equilibrium strategies are given by

$$F(x) = \frac{1}{\theta - \delta} \ln(1 + \frac{\theta - \delta}{v} x).$$

The expected equilibrium payoffs are

$$EU = \frac{v\theta(\theta - \delta) + v\theta\left(1 - e^{\theta - \delta}\right)}{(\theta - \delta)^2}.$$

Territorial contests with injury Similarly, consider an all-pay auction model of a territorial contest in which the outcome of the battle is not the result of the length of time that the respective rivals are willing to fight, but rather the intensity of effort x_1 and x_2, respectively, which they put forth in the fight. Assume that the cost to a player per unit of intensity of effort is 1, so that $\alpha = \beta = 1$, but also that each unit of intensity of effort put forth by player i in the battle imposes a cost on its rival (through injury, say), so that δ, $\theta > 0$. If the cost to the loser per unit of intensity of effort of the winner is greater than the cost to the winner per unit of intensity of effort of the loser, that is, $\theta > \delta > 0$, then $V = v + \gamma > 0$, $\alpha - \beta = 0$, and $\eta = \theta - \delta > 0$, so that case 2 of solution (4) applies. Symmetric equilibrium strategies take the same form as in the innovation game. If, on the other hand, winner and loser effort damages a rival symmetrically, so that $\theta = \delta > 0$, then the symmetric equilibrium mixed strategy is given by $F(x) = x/v$. This is different from the classic war of attrition where $\beta = \theta = 0$ and $\alpha = \delta = 1$, when case 3 of solution (4) with the exponential distribution applies. In the war of attrition the winning bid is not paid but only signals the willingness to expend that effort.

Litigation contests with fee shifting Baye et al. (2005) present a private values auction theoretic model of litigation environments to examine the effect of various fee shifting rules on the expenditure of legal effort. In that model, players compete in (quality normalized) expenditures on legal services, x_1 and x_2, with the player accruing the higher expenditure winning the case. All of the fee shifting rules examined by Baye et al. may be examined under complete information using the tools developed in Section 2. We examine two of these fee shifting rules here.

First under the Continental Rule, the loser of a legal contest pays his own legal expenditure and, additionally reimburses the winner a fraction $(1-\beta)$, $0 < \beta < 1$, of the winner's expenses. Under this system payoffs are

$$u_i(x_1, x_2) = \begin{cases} v - \beta x_i & \text{if} \quad x_i > x_j \\ -x_i - (1-\beta)x_j & \text{if} \quad x_i < x_j \end{cases}$$

so that $0 < \beta < \alpha = 1$, $\delta = 0 < \theta = (1-\beta)$. In this case, $V = v > 0$, $\alpha - \beta > 0$, and $\eta = 2(1-\beta) > 0$ so that case 1 of solution (4) applies, the distribution is

$$F(x) = \frac{1}{1-\beta}\left[1 - \left(\frac{v}{v + 2(1-\beta)x}\right)^{\frac{1}{2}}\right],$$

while the equilibrium payoff is $EU = -\gamma - \frac{\theta\alpha}{\eta}\frac{V}{\theta-\delta}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\theta-\delta}{\alpha-\beta}} - 1\right] + \frac{\theta}{\eta}V = -\frac{1-\beta}{\beta}\frac{v}{2} < 0.$

Baye et al.(2005) also examine a fee shifting rule in which the loser pays his own legal cost and reimburses the winner up to the level of the loser's own cost. The resulting contest, termed the Quayle Contest, satisfies $\alpha = 2$, $\beta = 1, \delta = -1$, and

$\theta = 0$. [4] Resulting payoffs are

$$u_i(x_1, x_2) = \begin{cases} v - x_i + x_j & \text{if} \quad x_i > x_j \\ -2x_i & \text{if} \quad x_j > x_i \end{cases}.$$

In this game $V = v > 0$, $\alpha - \beta \neq 0$, and $\eta \neq 0$, so that

$$F(w) = 2\left[1 - \left(\frac{v}{v + 2w}\right)^{\frac{1}{2}}\right]$$

and $EU = -\gamma - \frac{\theta\alpha}{\eta}\frac{V}{\theta-\delta}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\theta-\delta}{\alpha-\beta}} - 1\right] + \frac{\theta}{\eta}V = 0.$

Inequality aversion in a job tournament Consider two workers who compete in a job tournament in which their income is determined by relative efforts and who have inequality averse utility functions as given by a symmetric version of the Fehr-Schmidt (1999) model. Here, each player's utility of income is

$$u_i(y_i, y_j) = \begin{cases} y_i - b(y_i - y_j) & \text{if} \quad y_i > y_j \\ y_i & \text{if} \quad y_i = y_j \\ y_i - a(y_j - y_i) & \text{if} \quad y_i < y_j \end{cases}$$

where $0 < b < 1$ and $b < a$. Suppose the job tournament is a standard all-pay auction in which the player exerting the greater effort wins a prize valued at $\mu > 0$ and in which efforts are denoted $x_i \in [0, \mu]$. In this case, income as a function of effort is

$$y_i = \begin{cases} \mu - x_i & \text{if} \quad x_i > x_j \\ \mu/2 - x_i & \text{if} \quad x_1 = x_2 \\ -x_i & \text{if} \quad x_\iota < x_j \end{cases}$$

and, hence, utility as a function of effort is

$$u(x_1, x_2) = \begin{cases} \mu(1-b) - x_i(1-b) - bx_j & if \quad x_i > x_j \\ \mu/2 - x_i & if \quad x_i = x_j \\ -\mu a - x_i(1+a) + ax_j & if \quad x_i < x_j \end{cases}.$$

With these preferences the job contest clearly becomes a game with rank-order dependent spillovers. the loser's expenditure generates a *first-order negative spillover* effect on the winner and the winner's expenditure exerts a *second-order positive spillover* effect on the loser. Note that inequality aversion also generates first and

[4] We call this the "Quayle Contest" because Dan Quayle chaired the President's Council on Competitiveness, which proposed this mechanism as a fee shifting rule in order to avoid excessive litigation.

second order direct effects that are asymmetric; the loser suffers a greater marginal disutility form his effort than does the winner.[5]

Placing this in the context of (1), $v = \mu(1-b)$, $\gamma = \mu a$, $V = v + \gamma = \mu(1-b+a) \geq \mu > 0$, $\alpha = (1+a) > 0$, $\beta = (1-b) > 0, \theta = -a \leq 0$, $\delta = b \geq 0$, and $\eta = \alpha - \beta + \theta - \delta = (1+a) - ((1-b)) - (a) - (b) = 0$. Note first that from our Proposition 2, since $V > 0$ and $\eta = 0$, there can be no symmetric pure strategy Nash equilibrium.

Since $V > 0, \eta = 0, \alpha > 0, \beta > 0$, and $\alpha - \beta = a + b \geq 0$, from 4 if a nondegenerate symmetric mixed strategy equilibrium is to exist then it must be of one of the following two forms:

(i): If $\alpha - \beta = a + b = 0$,which implies $a = b = 0$ ("self-regarding preferences"), then from 4, $F(x) = x/\mu$ on the interval $[0, \mu]$, and each player earns expected equilibrium payoffs of 0.

(ii): If $\alpha - \beta = a + b > 0$, which implies that $a > 0$, then from 4 the unique symmetric equilibrium is

$$F(x) = \frac{1+a}{a+b}\left(1 - \exp\left(-\frac{a+b}{\mu(1+a-b)}x\right)\right) \text{ on } \left[0, \frac{\mu(1+a-b)}{a+b}\ln\frac{1+a}{1-b}\right]$$

and furthermore, each player earns expected equilibrium payoffs of

$$EU = \mu a\left[-1 + (1+a-b)\frac{(1-b)\ln\left(\frac{1-b}{1+a}\right) + a + b}{(a+b)^2}\right].$$

Loss aversion in a job tournament Consider again two workers who compete in a job tournament. Their income is determined by relative efforts, denoted $x_i \in [0, v]$, in a standard all-pay auction in which the player exerting the greater effort wins a prize valued at $\mu > 0$:

$$y_i = \begin{cases} \mu - x_i & \text{if} \quad x_i > x_j \\ \mu/2 - x_i & \text{if} \quad x_1 = x_2 \\ -x_i & \text{if} \quad x_\iota < x_j \end{cases}.$$

Now assume that players' preferences exhibit loss aversion so that utility as a function of income y is y if player i wins and λy if player i loses, where $\lambda > 1$. Hence, utility as a function of effort is

$$u_i(x_i, x_j) = \begin{cases} \mu - x_i & \text{if} \quad \mu \geq x_i > x_j \\ \mu/2 - x_i & \text{if} \quad x_i = x_j \\ -\lambda x_i & \text{if} \quad x_\iota < x_j \end{cases}.$$

In this case $v = \mu$, $\gamma = 0$, $V = v + \gamma = \mu > 0$, $\alpha = \lambda > 0$, $\beta = 1 > 0, \theta = \delta = 0$, and $\alpha - \beta = \eta = \alpha - \beta + \theta - \delta = \lambda - 1$. Hence, from 4

[5] This is also true in the case of preferences exhibiting loss aversion, which can also be examined in the context of our model.

$$F(x) = \frac{\lambda x}{v + (\lambda - 1)x}.$$

Regret in auctions Filiz-Ozbay and Ozbay(2007) recently studied overdissipation in auctions with anticipated regret. Utility in the first-price auction with symmetric prize v and with winner regret takes the form:

$$u_i(x_1, x_2) = \begin{cases} v - x_i - \mu(x_i - x_j) & \text{if} \quad x_i > x_j \\ 0 & \text{if} \quad x_i < x_j \end{cases}$$

where x_i is player $i's$ bid, $i = 1, 2$, and $\mu > 0$. Winner regret implies that the high bidder derives disutility from leaving money on the table (the difference between the winning and losing bid). Note that this is implies

$$u_i(x_1, x_2) = \begin{cases} v - x_i(\mu + 1) + \mu x_j & \text{if} \quad x_i > x_j \\ 0 & \text{if} \quad x_i < x_j \end{cases}$$

which converts the auction into a rank-order contest with a *positive first-order spillover* effect.

In this case, $V = v$, $\alpha = \theta = 0$, $\beta = (1 + \mu) > 0$, $\delta = -\mu$, and $\eta = \alpha - \beta + \theta - \delta = -1$. So, by Proposition 2 a symmetric pure strategy Nash equilibrium exists and is given by

$$x = \frac{V}{-\eta} = \frac{v}{1} = v.$$

There are no symmetric mixed strategy equilibria.

Filiz-Ozbay and Ozbay(2007) also examine loser regret in the first-price auction:

$$u_i(x_1, x_2) = \begin{cases} v - x_i & \text{if} \quad x_i > x_j \\ -\rho(v - x_j) & \text{if} \quad x_i < x_j \end{cases}$$

where $\rho > 0$.Clearly, this converts the auction into a rank-order contest with a *positive second-order spillover* effect. There is again only a pure strategy equilibrium at v.

The results of Section 2 may also be applied to extend the analysis to cover combined winner-loser regret for the first-price auction, where both *first and second order spillover* effects arise,

$$u_i(x_1, x_2) = \begin{cases} v - x_i(\mu + 1) + \mu x_j & \text{if} \quad x_i > x_j \\ -\rho(v - x_j) & \text{if} \quad x_i < x_j \end{cases},$$

in which case the pure strategy solution is $v/(1 + \rho)$.

These first-price results are readily extended to other auction formats, such as combined winner-loser regret in first-price all pay auctions

$$u_i(x_1, x_2) = \begin{cases} v - x_i(\mu + 1) + \mu x_j & \text{if} \quad x_i > x_j \\ -v\rho - x_i(\rho + 1) + \rho x_j & \text{if} \quad x_i < x_j \end{cases}$$

In this last case, $v+\gamma=(1+\rho)\,v$, $\alpha=(1+\rho)>0$, $\beta=(1+\mu)>0$, $\theta=-\rho$, $\delta=-\mu$,and $\eta=\alpha-\beta+\theta-\delta=(1+\rho)-(1+\mu)-\rho+\mu=0$. For $\rho\neq\mu$, so that $\alpha\neq\beta$, the case 3 of solution 4 applies. The unique mixed strategy equilibrium is

$$F(x)=\frac{1+\rho}{\rho-\mu}\left[1-\exp\left(-\frac{\rho-\mu}{1+\rho}\frac{x}{v}\right)\right],$$

and has

$$EU=v\rho+\rho\,(1+\rho)\,v\frac{(1+\mu)\ln\frac{1+\mu}{1+\rho}+\rho-\mu}{(\rho-\mu)^2}.$$

If $\rho=\mu$, so that $\alpha=\beta$, we get the standard APA form:

$$F\,(x)=x/v.$$

Hence total effort is the same under symmetric winner loser regret as in the standard all-pay auction. Thye $EU=\frac{3}{2}v\rho$.

ESS in the all-pay auction Consider the two player all-pay auction where each player i earns v minus his bid x_i if he wins and loses his bid x_i if he is outbid. The unique Nash equilibrium strategy is for each player to draw his bid from a uniform distribution on $[0,v]$. The (finite agent) ESS equilibrium (see Schaffer, 1988) corresponding to this two player game is a uniform distribution on $[0,2v]$ and implies overdissipation. To show this, note for an ESS equilibrium, each player maximizes the difference in payoffs. Thus if player i wins, his payoff is

$$v-x_i-(-x_j)=v-x_i+x_j$$

If player i loses $(x_i<x_j)$, his payoff is

$$-x_i-(v-x_j)=-v-x_i+x_j.$$

Hence, payoffs are given by

$$u_i(x_1,x_2)=\begin{cases} v-x_i+x_j & \text{if} \quad x_i>x_j \\ -v-x_i+x_j & \text{if} \quad x_i<x_j \end{cases}.$$

Within our parameterization, $V=2\nu>0$, $\beta=\alpha=-\theta=-\delta=1$, and consequently, $\alpha-\beta=0$ and $\eta=0$. The Nash equilibrium corresponding to this game gives the ESS equilibrium

$$F(x)=\frac{x}{2v}.$$

Thus there will be overdissipation, just as in the case of the Tullock game examined in Hehenkamp et al. (2004).

More generally, the relative payoffs for the linear case we considered are

$$u_i(x_1, x_2) = \begin{cases} \nu - \beta x_i - \delta x_j - [-\gamma - \alpha x_j - \theta x_i] & \text{if} \quad x_i > x_j \\ -\gamma - \alpha x_i - \theta x_j - [\nu - \beta x_j - \delta x_i] & \text{if} \quad x_i < x_j \end{cases}$$

This system can always be written as

$$u_i(x_1, x_2) = \begin{cases} \nu + \gamma - (\beta - \theta) x_i - (\delta - \alpha) x_j & \text{if} \quad x_i > x_j \\ -(\nu + \gamma) + (\delta - \alpha) x_i + (\beta - \theta) x_j & \text{if} \quad x_i < x_j \end{cases}$$

or

$$u_i(x_1, x_2) = \begin{cases} \nu - \beta x_i + \alpha x_j & \text{if} \quad x_i > x_j \\ -\nu - \alpha x_i + \beta x_j & \text{if} \quad x_i < x_j \end{cases}$$

Thus $V = 2v$, $\delta = -\alpha$, $\theta = -\beta$ and $\eta = 2(\alpha - \beta)$. The following Nash equilibria are ESS equilibria to the corresponding original parametrization:

1. Pure ESS equilibrium if $\beta > 0$, $\nu \geq 0$, $\alpha x \leq 0$

$$x = \frac{\nu}{\beta - \alpha}.$$

2. Mixed ESS equilibrium $\beta = 0$, $\nu > 0$.
subcase(i): $\alpha > 0$ on $[0, \infty)$

$$F(x) = 1 - (\frac{\nu}{\nu + \alpha x})^{\frac{1}{2}}$$

subcase (ii) $\alpha < 0$,on $[m, \infty)$

$$F(x) = 1 - (\frac{\nu + \alpha m}{\nu + \alpha x})^{\frac{1}{2}}$$

2. Mixed ESS equilibrium $\beta > 0$, $\nu > 0$, $\alpha > 0$, $\alpha \neq \beta$, on $[0, \frac{1}{\alpha - \beta}\left(\left(\frac{\alpha}{\beta}\right)^2 - 1\right)]$

$$F(x) = \frac{\alpha}{\alpha - \beta}\left[1 - \left(\frac{\nu}{\nu + (\alpha - \beta)x}\right)^{\frac{1}{2}}\right].$$

3. In case $\alpha = \beta$, the uniform distribution from the standard APA emerges again.

3.2 Price Competition

The basic Varian model The price setting models with spillovers that we examine are extensions of the basic Varian (1980) model, which can be expressed in our framework as follows. Two firms each have $U > 0$ uninformed (or loyal) consumers. In addition there are I informed consumers who buy from the lower price firm. Firms produce at zero marginal cost. Consumers purchase one unit of the good if it is available at a price less than or equal to their reservation price r, with the informed buying from the low-price firm and uninformed buying from the firm to which they are attached. Payoffs are

$$\pi_i(p_i, p_j) = \begin{cases} (I+U)\,p_i & \text{if } p_i < p_j \\ (I/2+U)\,p_i & \text{if } p_i = p_j \\ Up_i & \text{if } p_i > p_j \end{cases}$$

Define $x_i = r - p_i \geq 0$. We can then write the payoffs as

$$\pi_i(x_i, x_j) = \begin{cases} (I+U)\,r - (I+U)\,x_i & \text{if } x_i > x_j \\ (I/2+U)\,r - (I/2+U)\,x_i & \text{if } x_i = x_j \\ rU - Ux_i & \text{if } x_i < x_j \end{cases}$$

Note that in this case, $v = (I+U)\,r, \gamma = -rU, v+\gamma = rI > 0, \beta = (I+U), \delta = \theta = 0, \alpha = U > 0$, and $\eta = \alpha + \theta - \beta - \delta = U + 0 - (I+U) - 0 = -I < 0$. By case 1 of the solution 4, we have

$$F(x) = \frac{U}{I}\left(\frac{r}{r-x} - 1\right)$$

on $\left[0, r\left(1 - \frac{U}{I+U}\right)\right]$. Note that $G(p) = \Pr(P < p) = 1 - \Pr(P > p) = 1 - \Pr(x < r-p) = 1 - F(r-p)$, so that on $\left[r\frac{U}{I+U}, r\right]$

$$G(p) = 1 - \frac{U}{I}\left(\frac{r-p}{p}\right).$$

This is exactly the Varian (1980) result.

Price matching Consider a version of the Varian model in which firms employ price matching polices. As in the original model, two firms compete in price, there are I informed consumers who are aware of both prices in the market and U uninformed consumers attached to each firm who are aware only of the price at their own firm. Each consumer purchases one unit up to a choke price, r. Firms list prices and also promise to match the price of the rival to any informed customer that shows up at its firm. Suppose that a proportion $\mu \leq 1/2$ of the informed customers are willing to bear the cost of visiting high price firm and taking it up on its offer to match the better price. In this case the payoff function is

$$\pi_i(p_i, p_j) = \begin{cases} Up_i + \mu I p_j & \text{if } p_i > p_j \\ (I/2+U)\,p_i & \text{if } p_i = p_j \\ ((1-\mu)\,I + U)\,p_i & \text{if } p_i < p_j \end{cases}$$

As before, let $x_i = r - p_i$. Then the payoffs are

$$\pi_i(x_i, x_j) = \begin{cases} r\,(I\,(1-\mu) + U) - x_i\,((1-\mu)\,I + U) & \text{if } x_i > x_j \\ r\,(I/2+U) - x_i\,(I/2+U) & \text{if } x_i = x_j \\ r\,(U + I\mu) - Ux_i - I\mu x_j & \text{if } x_i < x_j \end{cases}$$

Suppose $\mu < 1/2$.[6] Then $V = v + \gamma = Ir(1-2\mu) > 0$, $\beta = ((1-\mu)I + U) > 0$, $\alpha = U > 0$, $\theta = I\mu > 0$, $\delta = 0$, $\alpha - \beta = -(1-\mu)I > 0$, and $\eta = \alpha - \beta + \theta - \delta = -I(1-2\mu) < 0$. Since the parameter configuration is such that case 1 of solution 4 applies, where it should be noted that the lower bound m of the distribution of x_i is zero, the symmetric equilibrium mixed strategy is

$$F(x) = \frac{U}{(1-\mu)I}\left(\left(\frac{r}{r-x}\right)^{\frac{(1-\mu)}{(1-2\mu)}} - 1\right).$$

Solving $F(x) = 1$, we find that the upper bound of the symmtric equilibrium distribution of x is

$$u = r\left(1 - \left(\frac{U}{((1-\mu)I + U)}\right)^{\frac{(1-2\mu)}{(1-\mu)}}\right).$$

Reference pricing Suppose that all consumers have a choke price of $r > 0$, but that there are three types of consumers: L loyals who inelastically purchase from firm i provided $p_i < r$; S shoppers who inelastically purchase from the firm charging the lowest price; and a measure of "relative bargain" seekers who have downward sloping demand given by

$$D_i = \lambda\frac{p_j}{p_i}.$$

The relative bargain seekers only purchase from the firm charging the lowest price. Payoffs to the two firms are

$$\pi_i(p_i, p_j) = \begin{cases} Lp_i & \text{if } p_i > p_j \\ (S/2 + L + \lambda/2)\, p_i & \text{if } p_i = p_j \\ \left(S + L + \lambda\frac{p_j}{p_i}\right) p_i & \text{if } p_i < p_j \end{cases}.$$

As before, let $x_i = r - p_i$. Then the payoffs become

$$\pi_i(x_i, x_j) = \begin{cases} r(S + L + \lambda) - x_i(S + L) - \lambda x_j & \text{if } x_i > x_j \\ r(S/2 + L + \lambda/2) - x_i(S/2 + L + \lambda/2) & \text{if } x_i = x_j \\ rL - Lx_i & \text{if } x_i < x_j \end{cases}$$

Here $v + \gamma = r(S + \lambda) > 0$, $\alpha = L > 0$, $\beta = S + L > 0$, $\theta = 0$, $\delta = \lambda$, $\alpha - \beta = -S < 0$ and $\eta = \alpha - \beta + \theta - \delta = L - S - L - \lambda = -(S + \lambda) < 0$. Again, this corresponds to case 1 solution 4 and the symmetric equilibrium distribution of x is given by

$$F(x) = \frac{L}{S}\left(\left(\frac{r}{r-x}\right)^{\frac{S}{S+\lambda}} - 1\right)$$

[6] Suppose $\mu = 1/2$. Then $V = v + \gamma = 0$, $\beta = ((1-\mu)I + U) > 0$, $\alpha = U > 0, \theta = I\mu > 0$, $\delta = 0$, and $\eta = \alpha - \beta + \theta - \delta = -I(1-2\mu) = 0$. Hence, by Proposition 2, the unique symmetric pure strategy equilibrium is $x = 0$,which implies $p_i = r$.

on $\left[0, r\left(1-\left(\frac{L}{S+L}\right)^{\frac{(S+\lambda)}{S}}\right)\right]$. As before,

$$\begin{aligned} G(p) &= 1-F(r-p) \\ &= 1-\frac{L}{S}\left(\left(\frac{r}{p}\right)^{\frac{S}{S+\lambda}}-1\right) \end{aligned}$$

The upper bound is clearly r. The lower bound is thus

$$r-u=r-r\left(1-\left(\frac{L}{S+L}\right)^{\frac{(S+\lambda)}{S}}\right)=r\left(\frac{L}{S+L}\right)^{\frac{(S+\lambda)}{S}}.$$

Notice that this distribution converges to the distribution of the Varian model as λ tends to zero.

4 Conclusion

This paper has introduced a parameterized framework for analyzing a large class of two player complete information rank-order contests with rank-order dependent spillovers. We have derived a closed form characterization for the complete set of symmetric equilibrium strategies of this class and have shown that these strategies take on only a small number of functional forms that depend on the parameters in a systematic and easily verified way. We have used the framework to formulate several new contests and to illustrate how several existing contests may be extended to allow for a greater generality in preferences and a broader range of strategic effects. The logarithmic mixed strategy solution, for example, appears to be novel to the literature. The framework has potential positive externalities to future research in the area as it provides a complete map of the general linear case. Possible extensions are to n-agent and incomplete information settings.

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5 Appendix

The appendix gives the proofs and explicit expressions for the expected payoffs. We start by providing the proofs and statements regarding the pure strategy equilibria.

5.1 Conditions for Pure Strategy Nash Equilibria

Before we exploit the affine pay-off structure, we first give a general result if the payoffs are left unspecified

$$EU(x_i, x_j) = \begin{cases} W(x_i, x_j) & \text{if } x_i > x_j \\ L(x_i, x_j) & \text{if } x_i < x_j \\ T(x_i, x_j) = \frac{1}{2}W(x_i, x_j) + \frac{1}{2}L(x_i, x_j) & \text{if } x_i = x_j \end{cases}.$$

Theorem 4 *There exists a symmetric pure strategy Nash equilibrium if and only if* $\exists x \in [0, \infty)$ *such that the following two conditions hold:*

$$\begin{aligned} T(x,x) &\geq W(y,x) \ \textit{for all } y \geq x \\ T(x,x) &\geq L(y,x) \ \textit{for all } 0 \leq y \leq x \end{aligned} \tag{5}$$

Proof. Note first that since

$$T(x,x) = \frac{1}{2}W(x,x) + \frac{1}{2}L(x,x),$$

the conditions in (5) imply

$$T(x,x) = W(x,x) = L(x,x).$$

($\Longleftarrow$) By hypothesis, there exists an $x \in [0, \infty)$ such that

$$\begin{aligned} T(x,x) &\geq W(y,x) \ \text{for all } y \geq x \\ T(x,x) &\geq L(y,x) \ \text{for all } y \leq x \end{aligned}$$

Hence, if player i plays the pure strategy $x_i = x$ when player j plays $x_j = x$, she earns a payoff of $U^* = T(x,x) = W(x,x) = L(x,x)$. The conditions in (5) imply that player i cannot gain by deviating from x, given that $x_j = x$.

($\Longrightarrow$) If (x, x) is a symmetric pure strategy Nash equilibrium, player i earns a payoff of $T(x,x)$ in this equilibrium. By way of contradiction, suppose there exists a $y \in [0, \infty)$ such that $y > x$ with $T(x,x) < W(y,x)$. Then player i could increase his payoff to $W(y,x) > T(x,x)$ by deviating from $x_i = x$ to $x_i = y$, a contradiction. Similarly, if there existed a $y \in [0, \infty)$ such that $y < x$ with $T(x,x) < L(y,x)$, player i could increase his payoff to $L(y,x) > T(x,x)$ by deviating from $x_i = x$ to $x_i = y$, a contradiction.

We conclude that the conditions in (5) are necessary and sufficient for the existence of a symmetric pure strategy Nash equilibrium. ■

Remark 5 *Note that the above theorem does not use linearity; the theorem is perfectly general in the sense that it applies to arbitrary* $W, L,$ *and* T *functions. Furthermore, it applies for arbitrary strategy spaces* $A \subseteq R$ *(compact or otherwise).*

We now turn to using the affine structure of (1). We will use the following abbreviations: $V \equiv v + \gamma$ and $\eta \equiv \alpha + \theta - \beta - \delta$. Following is a statement and proof of Proposition 1 from the main text.

Proposition 1 *Suppose $V \geq 0$, $A = [0, \infty)$ and W, L, and T are in the general linear class as described in (1). Then x is a symmetric pure strategy Nash equilibrium if and only if the following three conditions hold:*
(a) $\beta \geq 0$
(b) $x\alpha \leq 0$
(c) $V + \eta x = 0$.

Proof. ($\Longrightarrow$)By way of contradiction, suppose $x \in [0, \infty)$ is a symmetric pure strategy Nash equilibrium so that player i earns his equilibrium payoff of $U^* = T(x, x) = W(x, x) = L(x, x)$ at (x, x). If (a) did not hold, then player i could deviate to earn $W(x + \varepsilon, x) > U^* = W(x, x)$ (since $\beta < 0$ implies $W(x_i, x)$ is increasing in x_i), a contradiction. If (b) did not hold, then $x > 0$ and $\alpha > 0$, in which case player i could deviate to earn a payoff of $L(x - \varepsilon, x) > L(x, x) = L(x, x) = U^*$, (since $\alpha > 0$ implies $L(x_i, x)$ is decreasing in x_i), a contradiction. Finally, if (c) did not hold, then $V + \eta x = W(x, x) - L(x, x) \neq 0$, which contradicts the conditions in (5).

($\Longleftarrow$) Suppose conditions (a) through (c) hold. It is immediate that (c) implies $T(x, x) = W(x, x) = L(x, x)$; (a) implies $T(x, x) \geq W(y, x)$ for all $y \geq x$, and (b) implies $T(x, x) \geq L(y, x)$ for all $y \leq x$. By the Theorem, this implies that x is a symmetric pure strategy Nash equilibrium. Q.E.D. ∎

Proposition 2 from the main text is directly implied by Proposition 1. The following corollary is also an immediate consequence of Proposition 1.

Corollary 6 *Suppose $V > 0$, $A = [0, \infty)$ and W, L, and T are in the general linear class as described in (1). Then any symmetric pure strategy Nash equilibrium is interior and unique.*

5.2 Symmetric Mixed Strategy Equilibrium

Before we obtain the general form of the solution, we first characterize the continuity of the solution.

Lemma 7 *If there is an atom at some point $x \in [0, \infty)$, then x must be a symmetric pure strategy Nash equilibrium.*

Proof. If there is an atom of size $n(x) \in (0, 1)$ at some point x, it must be the case that $n(x)[W(x + \varepsilon, x) - T(x, x)] \leq 0$ (no incentive to raise the bid above x) and $n(x)[L(x - \varepsilon, x) - T(x, x)] \leq 0$ (no incentive to lower the bid below x) for small $\varepsilon > 0$. Since $n(x) > 0$ by hypothesis, this implies $[W(x + \varepsilon, x) - T(x, x)] \leq 0$ and $[L(x - \varepsilon, x) - T(x, x)] \leq 0$. This implies $T(x, x) = W(x, x) = L(x, x)$, and furthermore, given the linearity of W and L,

$$\begin{aligned} T(x, x) &\geq W(y, x) \text{ for all } y \geq x \\ T(x, x) &\geq L(y, x) \text{ for all } y \leq x. \end{aligned}$$

These are exactly the conditions (5) for a pure strategy solution from Theorem 4 and hence x must also be a pure strategy equilibrium point. ■

Lemma 8 *Suppose there is an atom of size $n(x) \in (0,1]$. Then $\beta \geq 0$, $\alpha x \leq 0$ and $\eta \leq 0$. Furthermore, if $V > 0$, the mass point is unique $x = -V/\eta$.*

Proof. By Lemma 7 and Propositions 1 and 2 part (b), these claims follow immediately. ■

Lemma 9 *Suppose $\alpha > 0$. Then in any non-degenerate symmetric mixed-strategy Nash equilibrium, the lower bound of the support is $m = 0$ if there are no mass points. With mass points and if $V > 0$, again necessarily $m = 0$. With mass points and if $V = 0$, then $m = 0$ if $\alpha > \eta/2$.*

Proof. The proof proceeds by way of contradiction, suppose the lower bound of the support of the equilibrium mixed-strategy is $m > 0$. The proof is divided into three parts a, b and c.
a) If F has no atoms, a player who bids m is sure to lose but nonetheless earns his "equilibrium" payoff of $U^* = -\gamma - \alpha m - \theta E_F[x]$. But a player who deviates to $m = 0$ earns $U^{**} = -\gamma - \theta E_F[x] > U^*$, a contradiction.
b) If there is a mass point of size q at m,

$$\begin{aligned} U^* &= qT(m,m) + (1-q)(-\gamma - \alpha m - \theta E_F[x|x>m]) \\ &= \frac{q}{2}V - \gamma + \frac{q}{2}\alpha m - \alpha m - \frac{q}{2}(\theta+\beta+\delta)m - (1-q)\theta E_F[x|x>m]. \end{aligned}$$

Deviating to playing zero gives instead

$$U^{**} = -\gamma - \theta q m - \theta(1-q)E_F[x|x>m].$$

The difference in payoffs thus reads

$$U^{**} - U^* = \frac{q}{2}\{-V + (-\theta + \alpha + \beta + \delta)m\} = q\alpha m > 0,$$

since if $V > 0$, then by Lemma 8 we know that $-V = \eta m$. Therefore it pays to deviate to playing zero.
c) If $V = 0$, then

$$U^{**} - U^* = \frac{q}{2}\{-\theta + \alpha + \beta + \delta\}m = q\alpha m - \frac{q}{2}\eta m.$$

It would pay to deviate and bid zero if $\alpha - \eta/2 > 0$. ■

If $V \geq 0$ and $\beta < 0$, then the support is unbounded above.

Lemma 10 *Suppose $V = v + \gamma \geq 0$ and $\beta < 0$. Then if a non-degenerate symmetric mixed-strategy Nash equilibrium exists, it must have an unbounded upper support.*

Proof. By way of contradiction, suppose that the upper bound of the support of the equilibrium mixed-strategy is $u < \infty$.
If F has no atoms, a player that bids u is sure to win and earn his equilibrium payoff of $U^* = v - \beta u - \delta E_F[x]$. But a player who deviates to $u' > u$ earns $U^{**} = v - \beta u' - \delta E_F[x_j] > U^*$ since $-\beta u' > -\beta u$, a contradiction. If there is a mass point p at u, the payoff at u is slightly more complicated

$$\widehat{U} = (1-p)\{v - \beta u - \delta E_F[x|x<u]\} + p\left\{\frac{v-\gamma}{2} - \frac{\alpha+\theta+\beta+\delta}{2}u\right\}.$$

But for $z >> u$ sufficiently large, again $\widetilde{U} = v - \beta z - \delta E_F[x_j] > \widehat{U}$. We conclude that $\beta < 0$ implies $u = \infty$. ■

The symmetric equilibrium mixed strategies are characterized in the Proposition 3, which is repeated here from the main text for convenience.

Proposition 3 *For the game with the affine payoff schedules W, L and T as defined in (1), suppose there exists a symmetric mixed strategy equilibrium with the equilibrium distribution $F(w)$. Furthermore, suppose $F(w)$ is at least once differentiable on the open subset (m,u) of the support; where $m \geq 0$ and possibly $u = \infty$. Then*

$$F(w) = \frac{\alpha}{\alpha-\beta}\left\{1 - \left[\frac{V+\eta m}{V+\eta w}\right]^{\frac{\alpha-\beta}{\eta}}\right\} + C\left[\frac{V+\eta m}{V+\eta w}\right]^{\frac{\alpha-\beta}{\eta}} \quad (6)$$

where $m \geq 0, 1 \geq C \geq 0, F(m) \geq 0, V \equiv v+\gamma \geq 0$ and $\eta \equiv \alpha+\theta-\beta-\delta$.

Proof. Since $F(w)$ has a density $f(w)$ on (m,u) by assumption, the expected payoff reads

$$EU(w) = \int_m^w W(w,x)f(x)dx + \int_w^u L(w,x)f(x)dx.$$

In equilibrium $EU(w)$ is constant on the subset (m,u) of the support, hence for $w\epsilon(m,u)$:

$$\frac{dEU(w)}{dw} = [V+\eta w]\, f(w) - \alpha + (\alpha-\beta)F(w) = 0. \quad (7)$$

The solution to this differential equation is known as (6); one easily verifies that (6) satisfies (7). As m is the lower bound of the support where $f(w)$ exists, $F(m) \geq 0$, which implies $C \geq 0$. For the reason that $F(w) \leq 1$, it also follows that $C \leq 1$ as $F(m) = C$. ■

Note that this only characterizes the solution by the first order condition. Not all parameter combinations do yield a well defined distribution function and the second order condition has to be verified. Below we provide an exhaustive description of all the configurations that do have a mixed strategy solution.

Remark 11 *Note that the sum $W+L$ is continuous, also at the points where the individual payoffs are discontinuous. Since at those points of discontinuity of W and L, still $W+L = 2T$. Moreover, the points of discontinuity are only along the* *diagonal in $\mathbb{R}^2_+$. This means if the strategy space is bounded one can invoke Theorem 6 of Dasgupta and Maskin (1986, I) to conclude there does always exist a symmetric equilibrium, possibly mixed. In our analysis we do, however, allow for an unbounded upper bound of the strategy space A. Until this point we considered the strategy space $A = [0, \infty)$. Thus the novel contribution of our paper is not only to provide the explicit solution for the general linear case, but it also covers cases not treated by the Dasgupta and Maskin theorem.*

The following subsections describe all parameter configurations that yield a mixed strategy equilibrium.

5.3 Configuration $\alpha \neq 0$ and $V > 0$.

We have to consider four cases distinguished by $\alpha = \beta$ or $\alpha \neq \beta$, and $\eta = 0$ or $\eta \neq 0$. We show that the solutions are as in (4) from the main text. By Lemma 8 we know that there can be only a single mass point as $V > 0$. In fact, note that the solutions (4) contain no mass point.

5.3.1 Case I. $\alpha \neq \beta$, $\eta \neq 0$, $V > 0$

There are three subcases depending on whether $\beta \gtreqless 0$.

Subcase I1. $\beta = 0$ This case is further subdivided into the cases $\alpha > 0$ and $\alpha < 0$. For each of these subcases we have to consider η positive and η negative separately.

If $\alpha > 0$, then there can be no mass point according to Proposition 2 and Lemma 7. By Lemma 9, $m = 0$ and from (3)

$$F(w) = 1 - \left(\frac{V}{V + \eta w} \right)^{\alpha/\eta}. \tag{8}$$

With $\eta > 0$, the upper bound $u = \infty$. While for $\eta < 0$

$$u = -V/\eta < \infty.$$

The expected payoff to a player in equilibrium is constant anywhere on the support. Hence the EU can be computed from picking any point in the support. In the case $\eta > 0$, it pays to pick the bid at the lower boundary 0 to determine the EU. But note that if $\eta > 0$ the distributions (8) have fat tails (vary regularly at infinity) so that not all moments exist; hence, we need an extra condition to be able to calculate the expected payoff (the sad loser game is an example when $\alpha = 0$, $\theta = \delta = 0$ and $\eta = 1$). If $\alpha/\eta - 1 > 0$ or $\delta > \theta$, the expected equilibrium payoff is bounded and reads

$$EU = -\gamma - \alpha * 0 - \theta \int_0^\infty x\alpha \left(\frac{V}{V + \eta x} \right)^{\alpha/\eta} \frac{1}{V + \eta x} dx = -\frac{\theta v + \delta\gamma}{\delta - \theta}.$$

Note that since the support is $[0, \infty)$, one can not deviate to bidding outside the support to gain.

In the case $\eta < 0$, it again pays to pick the bid at the lower boundary 0. The expected equilibrium payoff is again

$$EU = -\gamma - \alpha * 0 - \theta \int_0^u x\alpha \left(\frac{V}{V+\eta x}\right)^{\alpha/\eta} \frac{1}{V+\eta x} dx = -\frac{\theta v + \delta\gamma}{\delta - \theta}.$$

The support is $[0, u]$ and deviating by bidding above u does not improve the payoff, nor does it deteriorate the expected payoff. Hence the equilibrium satisfies the weak Nash requirement that one can not gain by deviating.

If $\alpha < 0$ and $\eta > 0$, there can be no mass point either by Proposition 2 and Lemma 7 as $\eta > 0$. But if $\alpha < 0$ and $\eta > 0$

$$\left(\frac{V+\eta w}{V+\eta m}\right)^{-\alpha/\eta} \geqslant 1$$

and hence (3) implies $F(w) \leq 0$, which cannot be the case. So no mixed strategy nor a pure strategy equilibrium exists in this case.

But $\alpha < 0$ and $\eta < 0$ is possible. In this case there may be a single mass point at $-V/\eta$. Suppose $-V/\eta$ is played with probability q while $m > -V/\eta$. Then the payoff to bidding at $-V/\eta$ is

$$\begin{aligned} EU(\frac{V}{-\eta}) &= qT\left(\frac{V}{-\eta}, \frac{V}{-\eta}\right) + (1-q)\left(-\gamma - \alpha\frac{V}{-\eta} - \theta E[y|y > \frac{V}{-\eta}]\right) \\ &= \frac{q}{2}\left[v - \gamma - (\delta + \alpha + \theta)\frac{V}{-\eta}\right] \\ &\quad + (1-q)\left(-\gamma - \alpha\frac{V}{-\eta} - \theta E[y|y > \frac{V}{-\eta}]\right). \end{aligned}$$

If one bids at m, one's payoff is

$$EU(m) = q\left(v - \delta\frac{V}{-\eta}\right) + (1-q)\left(-\gamma - \alpha m - \theta E[y|y > m]\right).$$

Note that by the continuity of $F(w)$ on the interval $[m, \infty)$, it is the case that $E[y|y > m] = E[y|y > \frac{V}{-\eta}]$. Thus

$$EU(m) - EU(\frac{V}{-\eta}) = -\alpha(1-q)\left(m - \frac{V}{-\eta}\right) > 0.$$

Hence there can be no mass point and a flat such that $m > -V/\eta$. Moreover for $\alpha < 0$ and $\eta < 0$ on any open interval (m, w) the solution (3) requires that $m > -V/\eta$ for $F(w)$ to be non-degenerate. It follows that if $\alpha < 0$ and $\eta < 0$ with $w \in [m, \infty)$ and for any $m > -V/\eta > 0$, the distribution

$$F(w) = 1 - \left(\frac{V + \eta m}{V + \eta w}\right)^{\alpha/\eta} \tag{9}$$

is a valid equilibrium. It pays to pick the bid at the lower boundary m to derive the expected payoff. Provided that $\alpha/\eta > 1$ the expected equilibrium payoff is

$$\begin{aligned} EU &= -\gamma - \alpha m - \theta \int_m^\infty x\alpha \left(\frac{V + \eta m}{V + \eta x}\right)^{\alpha/\eta} \frac{1}{V + \eta x} dx \\ &= -\frac{\theta v + \delta\gamma + \alpha\delta m}{\delta - \theta}. \end{aligned}$$

Since the support is $[m, \infty)$, note that one can not deviate to bidding below m to gain as this would in fact deteriorate the expected payoff.

Subcase I2. $\beta < 0$ Note that if $\beta < 0$ there can be no mass point by Proposition 2. The solution, if it exists, must read

$$F(w) = \frac{\alpha}{\alpha - \beta} \left(1 - \left(\frac{V + \eta m}{V + \eta w}\right)^{\frac{\alpha - \beta}{\eta}}\right).$$

By Lemma 10 $u = \infty$. Thus from solution (3) and if $(\alpha - \beta)/\eta > 0$

$$F(u) = \frac{\alpha}{\alpha - \beta} \neq 1.$$

If $(\alpha - \beta)/\eta < 0$, then $F(u) = \pm\infty$. Thus $F(w)$ is not a valid cdf and there does not exist a symmetric mixed strategy equilibrium.

Subcase I3, $\beta > 0$ First consider $\alpha < 0$. If $\eta > 0$ there can be no mass point by Lemma 8. But no mixed strategy equilibrium can exist as at the upper bound u, the conjectured solution (3) implies

$$\frac{\beta}{\alpha} = \left(\frac{V + \eta m}{V + \eta u}\right)^{\frac{\alpha - \beta}{\eta}}.$$

The LHS is strictly negative by assumption, while the RHS is necessarily non-negative (this includes the case $u = \infty$), a contradiction. But if $\eta < 0$ there might be a mass point of size C. In this case at the upperbound u, the conjectured solution (3) implies

$$\frac{\beta}{\alpha} = \left[1 - \frac{\alpha - \beta}{\alpha} C\right] \left(\frac{V + \eta m}{V + \eta u}\right)^{\frac{\alpha - \beta}{\eta}}. \tag{10}$$

The LHS is negative by assumption. Moreover for $F(w)$ to be non-decreasing, it is required that

$$-\frac{\alpha}{\alpha - \beta} + C < 0$$

so that $1-\frac{\alpha-\beta}{\alpha}C>0$. Thus the RHS of (10) is necessarily non-negative, again a contradiction.

Next, we turn to $\alpha>0$. Note that if $\alpha>0$ there can be no mass point by Proposition 2. In this case the solution (3) reads

$$F(w)=\frac{\alpha}{\alpha-\beta}\left[1-\left(\frac{V}{V+\eta w}\right)^{\frac{\alpha-\beta}{\eta}}\right]. \tag{11}$$

Note that $m=0$ by Lemma 9. Moreover, the upperbound reads

$$u=\frac{V}{\eta}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\eta}{\alpha-\beta}}-1\right].$$

Note that the upper bound u is finite and positive for all possible configurations of η and α, β. The density is

$$f(w)=\frac{\alpha}{V}\left(\frac{V+\eta w}{V}\right)^{\frac{\beta-\alpha}{\eta}-1}.$$

The EU can be computed from payoff to a bid at the lower boundary 0:

$$\begin{aligned} EU &= -\gamma-\alpha*0-\theta\int_0^u x\frac{\alpha}{V}\left(\frac{V+\eta x}{V}\right)^{\frac{\beta-\alpha}{\eta}-1}dx \\ &= -\gamma-\frac{\theta\alpha}{\eta}\frac{V}{\theta-\delta}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\theta-\delta}{\alpha-\beta}}-1\right]+\frac{\theta}{\eta}V. \end{aligned}$$

Also note that since the support is $[0,u]$, one can not deviate to bidding above u since due to $\beta>0$ this would lower the payoff from winning.[7]

5.3.2 Case II. $\eta\neq 0,\alpha=\beta;V>0$

If $\alpha=\beta$, it is immediate that $\alpha=\beta\neq 0$. If $\beta>0$, there can be no mass point by Proposition 2 and Lemma 7 since $\alpha>0$. To obtain the expression for the equilibrium

[7] If $(\alpha-\beta)/\eta=1$, i.e. $\theta-\delta=0$, then by taking limits of this solution, or by direct integration, the EU is

$$\frac{\theta\alpha}{\eta}\frac{V}{\eta}\ln\left(1+\beta/\eta\right)+\frac{\theta\alpha}{\eta}V\frac{1}{\beta+\eta}.$$

cdf in this particular case, consider solution (3) with $C = 0$ and take limits:

$$\begin{aligned}
\lim_{(\alpha-\beta)\longrightarrow 0} F(x) &= \alpha \lim_{(\alpha-\beta)\longrightarrow 0} \frac{1-\left(\frac{V+(\theta-\delta)m}{V+(\theta-\delta)x}\right)^{\frac{\alpha-\beta}{\theta-\delta}}}{\alpha-\beta} \\
&= -\frac{\alpha}{\theta-\delta}\ln\left(\frac{V+(\theta-\delta)m}{V+(\theta-\delta)x}\right)\lim_{(\alpha-\beta)\longrightarrow 0}\left(\frac{V+(\theta-\delta)m}{V+(\theta-\delta)x}\right)^{\frac{\alpha-\beta}{\theta-\delta}} \\
&= -\frac{\alpha}{\theta-\delta}\ln\left(\frac{V+(\theta-\delta)m}{V+(\theta-\delta)x}\right),
\end{aligned}$$

where we used l'Hôpital's rule in the second step. To further characterize the solution, note that $\alpha > 0$ implies by Lemma 9 $m = 0$. Thus

$$F(w) = \frac{\alpha}{\theta-\delta}\ln\left(\frac{V+(\theta-\delta)x}{V}\right)$$

on $[o, \frac{V}{\theta-\delta}\left(\exp\left(\frac{\theta-\delta}{\alpha}\right)-1\right)]$. Furthermore, note that $f(0) = \alpha/V$ and $f(u) = \alpha/V \exp\left(\frac{\theta-\delta}{\alpha}\right) > 0$. Thus, $F(w)$ is well defined regardless the sign of $\theta - \delta$. Mopreover, since $\beta > 0$, it does not pay to bid above the upper bound, as doing so increases costs but not the probability of winning.

The EU reads (computed from a bid at u)

$$\begin{aligned}
EU &= v - \beta\left(\frac{v+\gamma}{\theta-\delta}\left(\exp\left(\frac{\theta-\delta}{\alpha}\right)-1\right)\right) \\
&\quad -\delta\int_0^{\frac{v+\gamma}{\theta-\delta}\left(\exp\left(\frac{\theta-\delta}{\alpha}\right)-1\right)} w\frac{\alpha}{v+\gamma+w(\theta-\delta)}dw \\
&= \frac{(\theta-\delta)(v\theta+\gamma\delta)+\alpha\theta\left(1-e^{\frac{\theta-\delta}{\alpha}}\right)(v+\gamma)}{(\theta-\delta)^2}.
\end{aligned}$$

Now consider the case $\alpha = \beta < 0$. It is immediate that there can be no mass point according to Proposition 2 and Lemma 7 since $\beta < 0$. But

$$F(x) = \frac{\alpha}{\theta-\delta}\ln\left(\frac{V+(\theta-\delta)x}{V+(\theta-\delta)m}\right)$$

cannot be a Nash solution since at $F(u) = 1$, where

$$u = \left[\frac{V}{\theta-\delta}+m\right]e^{(\theta-\delta)/\alpha} - V$$

one always has a strict incentive to raise the bid above u as $\beta < 0$.

5.3.3 Case III. $\eta = 0,\ \alpha \neq \beta;\ V > 0$

Since $\eta = 0$, there can be no mass point by Proposition 2 and Lemma 7. Proceeding similarly as in the Case II, we obtain from (3)

$$\begin{aligned}\lim_{\eta \longrightarrow 0} F(x) &= \frac{\alpha}{\alpha-\beta}\left[1 - \lim_{\eta/(\alpha-\beta)\longrightarrow 0}\left(1 - \frac{(\alpha-\beta)(x-m)}{V+\eta x}\frac{\eta}{\alpha-\beta}\right)^{\frac{\alpha-\beta}{\eta}}\right] \\ &= \frac{\alpha}{\alpha-\beta}\left[1 - e^{-\frac{\alpha-\beta}{V}(x-m)}\right].\end{aligned}$$

The density then reads

$$f(x) = \frac{\alpha}{V}e^{-\frac{\alpha-\beta}{V}(x-m)}.$$

For this to be a valid density, note that $\alpha > 0$ is necessary. This implies by Lemma 9 $m = 0$. Thus the final solution reads

$$F(x) = \frac{\alpha}{\alpha-\beta}\left[1 - e^{-\frac{\alpha-\beta}{V}x}\right].$$

There are then two subcases, corresponding to $\beta = 0$ and $\beta > 0$. First, suppose $\beta = 0$, then

$$F(w) = 1 - e^{-\frac{\alpha}{V}w},$$

on $[0, \infty)$. Moreover,

$$EU = -\gamma - \frac{\theta}{\alpha}V.$$

Since the support is $[0, \infty)$,one can not deviate to bidding outside the support to gain.

Secondly suppose that $\beta > 0$. Then the support of the distribution is bounded: $w\epsilon \left[0, \frac{V}{\alpha-\beta}\ln\frac{\alpha}{\beta}\right]$. The expected equilibrium payoff from bidding zero is

$$\begin{aligned}EU &= -\gamma - \alpha * 0 - \theta \int_0^{\frac{v+\gamma}{\alpha-\beta}\ln\frac{\alpha}{\beta}} w\frac{\alpha}{v+\gamma}e^{-\frac{\alpha-\beta}{v+\gamma}w}dw \\ &= -\gamma - \theta(v+\gamma)\frac{\beta\ln\left(\frac{\beta}{\alpha}\right) + \alpha - \beta}{(\alpha-\beta)^2}.\end{aligned}$$

Since $\beta > 0$, it does not pay to bid above the upper bound u, as doing so increases costs but not the probability of winning. We conclude that F is an equilibrium.

Note that $\beta < 0$ cannot yield a valid solution as this would imply that the distribution $F \leq \alpha/(\alpha-\beta) < 1$ (recall $\alpha > 0$).

5.3.4 Case IV. $\eta = 0$, $\alpha = \beta$, $V > 0$

Note that necessarily $\alpha > 0$, as otherwise there is a strict incentive to bid as much as is possible. So there can be no mass points and $m = 0$.

Either by taking limits from the cases II or III, one obtains (using III)

$$\begin{aligned} \lim_{(\alpha-\beta)\longrightarrow 0} F(x) &= \alpha \lim_{(\alpha-\beta)\longrightarrow 0} \frac{1-\exp\left(-\frac{\alpha-\beta}{V}x\right)}{\alpha-\beta} \\ &= \lim_{(\alpha-\beta)\longrightarrow 0} \left(\frac{\frac{x}{V}\exp\left(-\frac{\alpha-\beta}{V}x\right)}{1} \right) = \frac{\alpha}{V}x. \end{aligned}$$

Where again we used l'Hôpital's rule. Obviously, $\alpha > 0$, $u = V/\alpha$ and

$$EU = -\gamma - \theta \frac{V}{2\alpha}.$$

Similarly, starting from the case II, one obtains the same result since by l'Hôpital's rule

$$\lim_{(\theta-\delta)\longrightarrow 0} F(x) = \alpha \lim_{(\theta-\delta)\longrightarrow 0} \frac{\alpha}{\theta-\delta} \ln\left(\frac{V+(\theta-\delta)x}{V}\right) = \alpha\frac{x/V}{1}.$$

Since $\beta > 0$, it does not pay to bid above the upper bound, as doing so increases costs but not the probability of winning. We conclude that F is an equilibrium.

5.4 Configuration $\alpha \neq 0$ and $V = 0$.

We have to consider four cases distinguished by $\alpha = \beta$ or $\alpha \neq \beta$, and $\eta = 0$ or $\eta > 0$. The solution (3) now specializes to

$$F(w) = \frac{\alpha}{\alpha-\beta}\left\{1-\left[\frac{m}{w}\right]^{\frac{\alpha-\beta}{\eta}}\right\} + C\left[\frac{m}{w}\right]^{\frac{\alpha-\beta}{\eta}} \tag{12}$$

We will show that the solutions for the specific configurations are then as follows

$$F(w) = \begin{cases} 1-\left[\frac{m}{w}\right]^{\alpha/\eta} + C\left[\frac{m}{w}\right]^{\alpha/\eta} & \text{if} \quad \eta \neq 0; \alpha \neq 0;\ \beta = 0 \\ \text{no mixed strategy exists} & \text{if} \quad \eta \neq 0; \alpha \neq 0; \alpha-\beta = 0 \\ \text{no mixed strategy exists} & \text{if} \quad \eta = 0; \alpha \neq 0; \alpha-\beta \neq 0 \\ \text{no mixed strategy exists} & \text{if} \quad \eta = 0; \alpha \neq 0; \alpha-\beta = 0 \end{cases}. \tag{13}$$

5.4.1 Case I. $\alpha \neq \beta$, $\eta \neq 0; V = 0$

Before we discuss the specific solutions for this case, first note that because $V = 0$ while $\eta \neq 0$, part (a) of Proposition 2 implies that there can at most be a single mass point. Moreover because $\eta \neq 0$, this mass point is necessarily at zero!

There are three subcases depending on whether $\beta \gtreqless 0$.

Subcase I1. $\beta = 0$ The solution from (12) is

$$F(w) = 1 - \left[\frac{m}{w}\right]^{\alpha/\eta} + C\left[\frac{m}{w}\right]^{\alpha/\eta}$$

and where C is the size of the mass point at zero. The density reads

$$f(w) = (1 - C)\frac{\alpha}{\eta}\left[\frac{m}{w}\right]^{\alpha/\eta}\frac{1}{w}. \tag{14}$$

Note that the density is only well defined if $\alpha/\eta > 0$. Moreover, the lower bound of the continuous part must be strictly positive, i.e. $m > 0$.

This case is further subdivided into the cases $\alpha > 0$ and $\alpha < 0$. For each of these subcases we have to consider η positive and η negative separately.

Consider the case $\alpha > 0$. For $m > 0$ and $C = 0$ one gains from bidding below m, since this lowers the expenditures in case of losing. Hence all mass will be concentrated at zero and there only exists a pure strategy equilibrium. For $m > 0$ and $1 > C > 0$, there may exist a mixed strategy solution as well. Let $q = C$ be the mass at zero. Consider the payoff from bidding at zero

$$\begin{aligned} & qT(0,0) + (1-q)\{v - \theta E[x|x>m]\} \\ = \;& q\left(\frac{v}{2} - \frac{\gamma}{2}\right) + (1-q)\{v - \theta E[x|x>m]\} \\ = \;& v - (1-q)\theta E[x|x>m], \end{aligned}$$

as $v = -\gamma$. Bidding at m yields

$$\begin{aligned} & qW(m,0) + (1-q)\{v - \alpha m - \theta E[x|x>m]\} \\ = \;& q\{v - \delta m\} + (1-q)\{v - \alpha m - \theta E[x|x>m]\} \\ = \;& v - (1-q)\alpha m - q\delta m - (1-q)\theta E[x|x>m]. \end{aligned}$$

In equilibrium the two bids have to yield equal expected payoff in expectation. This will be the case if

$$-(1-q)\alpha m - q\delta m = 0$$

or

$$q = \frac{\alpha}{\alpha - \delta} < 1.$$

This requires that $\delta < 0$ and $\eta > 0$ for the density to make sense. Thus a mixed strategy solution with some mass at zero may exist. But some positive mass at zero is necessary! This is different when $\alpha < 0$.

Next take the case $\alpha < 0$. For $m > 0$ one now would lose from bidding below m rather than at or above m, since this lowers the revenues in case of losing (while the prospects of winning do not improve). Thus $C = 0$ and

$$F(w) = 1 - \left[\frac{m}{w}\right]^{\alpha/\eta}$$

is an equilibrium if $\eta < 0$ does hold as well. Note that this distribution exhibits fat tails. The expected payoff is bounded if $\alpha/\eta > 1$.

Does there also exist an equilibrium with $\alpha, \eta < 0$ while $C > 0$? It seems that the following is a possibility. Let q be the mass at zero. The payoff from bidding at zero is

$$\begin{aligned}
& qT(0,0) + (1-q)\{v - \theta E[x|x > m]\} \\
= \; & q\left(\frac{v}{2} - \frac{\gamma}{2}\right) + (1-q)\{v - \theta E[x|x > m]\} \\
= \; & v - (1-q)\,\theta E[x|x > m],
\end{aligned}$$

while bidding at m yields

$$\begin{aligned}
& qW(m,0) + (1-q)\{v - \alpha m - \theta E[x|x > m]\} \\
= \; & q\{v - \delta m\} + (1-q)\{v - \alpha m - \theta E[x|x > m]\} \\
= \; & v - (1-q)\,\alpha m - q\delta m - (1-q)\,\theta E[x|x > m].
\end{aligned}$$

In equilibrium the two bids have to yield equal expected payoff in expectation, which is the case if

$$q = \frac{\alpha}{\alpha - \delta} < 1. \tag{15}$$

This further requires $\delta > 0$ and $\alpha - \delta < 0$. Note that it does not pay to bid at a point like $m/2$ between zero and m either as this would yield (using (15)):

$$\begin{aligned}
& qW(m/2,0) + (1-q)\{v - \alpha m/2 - \theta E[x|x > m]\} \\
= \; & q\{v - \delta m/2\} + (1-q)\{v - \alpha m/2 - \theta E[x|x > m]\} \\
= \; & v - (1-q)\,\theta E[x|x > m],
\end{aligned}$$

which is just the payoff from bidding at zero.

The equilibrium distribution with the mass point is

$$F(w) = 1 - \left[\frac{m}{w}\right]^{\alpha/\eta} + q\left[\frac{m}{w}\right]^{\alpha/\eta}.$$

The EU can be computed from the payoff to a bid at 0 provided that $\alpha/\eta > 1$:

If there is a mass point

$$\begin{aligned}
EU &= v - (1-q)\,\theta E[x|x > m] \\
&= v - (1-q)\,\theta \int_m^\infty w \frac{\alpha}{\eta}\left[\frac{m}{w}\right]^{\alpha/\eta} \frac{1}{w} dw \\
&= v + \frac{\delta}{\alpha - \delta}\frac{\alpha\theta}{\delta - \theta} m.
\end{aligned}$$

Without a mass point for the case $\alpha < 0$

$$\begin{aligned}
EU &= v - \alpha m - \theta E[x] \\
&= v - \frac{\alpha\delta}{\delta - \theta} m.
\end{aligned}$$

If $\eta > 0$ and $\alpha < 0$ there is only a pure strategy equilibrium at zero; the density (14) would not be well defined as $\alpha/\eta < 0$.

Subcase I2. $\beta < 0$ Note that if $\beta < 0$ there can be no mass point by Proposition 2 part (a). Then by (12)

$$F(w) = \frac{\alpha}{\alpha - \beta} \left\{ 1 - \left[\frac{m}{w} \right]^{\frac{\alpha-\beta}{\eta}} \right\}.$$

By Lemma 10, $u = \infty$. But then

$$F(u) = \frac{\alpha}{\alpha - \beta} \neq 1.$$

Thus $F(.)$ is not a valid cdf and there does not exist a symmetric mixed strategy equilibrium, nor can there exist a pure strategy equilibrium since $\beta < 0$.

Subcase I3, $\beta > 0$ In this case the conjectured solution reads

$$F(w) = \frac{\alpha}{\alpha - \beta} \left\{ 1 - \left[\frac{m}{w} \right]^{\frac{\alpha-\beta}{\eta}} \right\} + C \left[\frac{m}{w} \right]^{\frac{\alpha-\beta}{\eta}}.$$

First consider $\alpha < 0$. First consider the case of $C = 0$. Thus the equilibrium distribution is

$$F(w) = \frac{\alpha}{\alpha - \beta} \left\{ 1 - \left[\frac{m}{w} \right]^{\frac{\alpha-\beta}{\eta}} \right\}.$$

But then it is impossible for $F(u) = 1$, since $\frac{\alpha}{\alpha-\beta} < 1$, so this cannot be an equilibrium. The same argument with $C > 0$ shows that there is no mixed strategy equilibrium in this case either.

Next consider $\alpha > 0$. If there is no mass and $m > 0$, the payoff from bidding at the lower bound is

$$v - \alpha m - \theta E[x].$$

But this is less than the payoff from bidding at zero: $v - \theta E[x]$ as $\alpha > 0$. A similar argument applies in case there is some mass $q < 1$ placed at zero. In that case relocating all mass to zero improves the payoff by $(1 - q)\alpha m$. Thus with $\alpha > 0$ no mixed strategy equilibrium exists. But a pure strategy at $x = 0$ does exist.

5.4.2 Case II. $\eta \neq 0, \alpha = \beta; V = 0$

Before we discuss the specific solutions for this case, first note that because $V = 0$ while $\eta \neq 0$, part (a) of Proposition 2 implies that there can at most be a single mass point. Moreover because $\eta \neq 0$, this mass point is necessarily at zero!

As $\alpha = \beta$, it is immediate that $\alpha = \beta \neq 0$. The solution (12) implies in this case that

$$F(w) = \frac{\alpha}{\theta - \delta} \ln\left(\frac{w}{m}\right) + C.$$

Thus if there is some mass, there is a flat until some $m > 0$. If $C = 0$, the support starts necessarily at some positive value $m > 0$.

To further characterize the solution, note that $\alpha > 0$ requires that $\theta - \delta > 0$. The upper bound of the support is

$$u = me^{(\theta-\delta)(1-C)/\alpha}.$$

Furthermore, note that the density on $(m, u]$ is

$$f(w) = \frac{\alpha}{\theta - \delta} \frac{1}{w}.$$

Can there exist a mass point? If $C > 0$, the payoff to bidding at zero is

$$\begin{aligned} U^* &= CT(0,0) + (1-C)(-\gamma - \alpha.0 - \theta E[x|x \geq m]) \\ &= \frac{C}{2}(v - \gamma) + (1-C)(-\gamma - \theta E[x|x \geq m]) \\ &= -\gamma - (1-C)\theta E[x|x \geq m]. \end{aligned}$$

If one bids at m, the payoff is

$$\begin{aligned} U^{**} &= C(v - \beta m - \delta.0) + (1-C)(-\gamma - \alpha m - \theta E[x|x \geq m]) \\ &= -\gamma - \alpha m - (1-C)\theta E[x|x \geq m]. \end{aligned}$$

Hence

$$U^{**} - U^* = -\alpha m < 0.$$

Thus $1 > C > 0$ cannot be a valid Nash equilibrium.

If $C = 0$ and $m > 0$, does one have an incentive to relocate mass below m? If one bids at m one is expected to make

$$W^* = -\gamma - \alpha m - \theta E[x]$$

as one is sure to loose. By relocating to bidding at $z < m$, one makes

$$W^{**} = -\gamma - \alpha z - \theta E[x].$$

Since $\alpha > 0$, this improves the situation as $W^{**} > W^*$. Hence there does not exist a mixed strategy solution that is a Nash equilibrium.

Now consider the case $\alpha = \beta < 0$. It is immediate that there can be no mass point according to Proposition 2 and Lemma 7 since $\beta < 0$. But

$$F(x) = \frac{\alpha}{\theta - \delta} \ln\left(\frac{x}{m}\right)$$

cannot be a Nash solution since at $F(u) = 1$, where

$$u = me^{(\theta-\delta)/\alpha}$$

one always has a strict incentive to raise the bid above u as $\beta < 0$.

5.4.3 Case III. $\eta = 0$, $\alpha \neq \beta$; $V = 0$

Since $\eta = 0$ and $V = 0$ imply that the differential equation (7) reads

$$F(x) = \frac{\alpha}{\alpha - \beta}.$$

Thus no mixed strategy solution exists for this configuration. But there exist pure strategy equilibria if $\beta \geq 0$ (which is unique if $\alpha > 0$).

5.4.4 Case IV. $\eta = 0$, $\alpha = \beta$, $V = 0$

The differential equation (7) in this case reads

$$-\alpha = 0$$

and hence no mixed strategy solution exists. Pure strategy equilibria exist, though, if $\alpha = \beta \geq 0$.

5.5 Configuration $\alpha = 0$ Mixed Strategy Equilibria

The solution (3) reduces in this case to

$$F(w) = C\left[\frac{V + \eta w}{V + \eta m}\right]^{\frac{\beta}{\eta}}. \tag{16}$$

This can always be written as

$$F(w) = D\left(V + \eta w\right)^{\beta/\eta}, \ D > 0. \tag{17}$$

The solution in this form also directly emerges from solving the differential equation (7) with α restricted to zero, as

$$\frac{f}{F} = \frac{\beta}{V + \eta w}$$

implies

$$\ln F = \frac{\beta}{\eta}\ln\left(V + \eta w\right) + \ln D,$$

and $D > 0$. There are three specific cases $\eta = \beta = 0$, $\eta = 0$ & $\beta \neq 0$ and $\eta \neq 0$. We find that in the first case no mixed strategy equilibrium exists; in the second case any distribution on $\mathbb{R}^+$ is a solution; and for the third case (17) is a solution if $V = 0$.

5.5.1 Case I. $\eta = 0$ & $\beta \neq 0$

We first investigate the case $V = 0$, then the case with $V > 0$.

subcase I.1 $V = 0$ From Proposition 2 part (a) we have that if $V = 0$, $\beta > 0$ and $\alpha = \eta = 0$, then any $x \in [0, \infty)$ is a symmetric pure strategy Nash equilibrium. One shows that $T(s, s) = v - \theta s$ and since $\beta > 0$, there is no incentive to overbid s, while underbidding does not improve one's payoff since $L(s - \varepsilon, s) = v - \theta s$ for $\varepsilon > 0$.

Does there also exist a mixed strategy equilibrium in which one mixes over some of these pure strategy equilibria in the case that $V = 0$? Suppose (x, z) are two pure strategy equilibria with $x < z$. Let q be the probability by which you play x and let p be the probability by which the opponent plays x. Your payoff in this case reads

$$\begin{aligned} EU &= q\left(p\left(v - \theta x\right) + \left(1 - p\right)\left(v - \theta z\right)\right) \\ &\quad + \left(1 - q\right)\left(p\left(v - \theta x - \beta\left(z - x\right)\right) + \left(1 - p\right)\left(v - \theta z\right)\right) \\ &= v + p\theta\left(z - x\right) - z\theta - \left(1 - q\right)p\beta\left(z - x\right) \\ &= v - \theta z + p\delta\left(z - x\right) + qp\beta\left(z - x\right). \end{aligned}$$

It follows that for any $p\epsilon(0, 1]$ you have a strict incentive to choose $q = 1$ as $z - x > 0$. In the symmetric equilibrium there are only two corner solutions $p = q = 1$ or $p = q = 0$. Thus a Bernoulli mixed strategy equilibrium does not exist. This can be generalized to any discrete distribution function.

subcase I.2 $V > 0$ Next, consider the cases with $V > 0$. From Proposition 2 part (b) we immediately have that if $V > 0$ and $\eta = 0$, there does not exist a symmetric pure strategy solution.

Does there exist a mixed strategy solution where the mixing is by a continuous distribution function? For the continuous mixture (7) must hold. Note that in this case (7) only makes sense if both $V > 0$ and $\beta > 0$, since $V = 0$ implies $F = 0$ as $\beta \neq 0$; $V > 0$ implies $\beta > 0$ as $f/F \geq 0$ necessarily. Either by taking limits as $\eta \to 0$, or by directly solving the differential equation $f/F = \beta/V$, one finds

$$F(w) = \exp\left(\frac{\beta}{V}w - Q\right)$$

and where $Q \geq 0$. Let u be the upper bound of the support, i.e. $F(u) = 1$, i.e. $u = QV/\beta < \infty$. In case there is a mass point at u, $u < QV/\beta$. In any case, $F(w) > 0$ for all $w \geq 0$, so there is at least one mass point. But this contradicts Proposition 2 part (b) that holds if $V > 0$ and $\eta = 0$, there is no symmetric pure strategy solution. In combination with Lemma 7, this means there can be no mass point. Thus $\exp\left(\frac{\beta}{V}w - Q\right)$ cannot be a valid equilibrium distribution.

Remark 12 *Note, though, that if the strategy space were restricted as in Dasgupta and Maskin (1986, I) to say $A = [0, u]$, then (u, u) would be a valid pure strategy equilibrium.*

5.5.2 Case II. $\eta = \beta = 0$

In this case $\theta = \delta \neq 0$ and own actions only influence the possibility of winning, but have no further effect on the own payoff.

subcase II.1 $V > 0$ If $V > 0$ and regardless the sign of $\theta = \delta$, both agents want to call out the highest possible number to ensure a win, and no Nash equilibrium exists (a win secures $v > -\gamma$, while the variable payoff parts cannot be influenced through own actions).

subcase II.2 $V = 0$ If $V = 0$, though, winning or losing gives the same fixed payoff part and we know from Propostion 2 that any $x \geq 0$ is a pure strategy equilibrium. Moreover note that for $x < z$ any of the four combinations (x, x), (x, z), (z, z), and (z, z) is a pure strategy equilibrium. Randomization over these outcomes leads to a symmetric mixed strategy equilibrium for any Bernoulli distribution. Let q be the probability by which you play x and let p be the probability by which the opponent plays x. Your payoff in this case reads

$$\begin{aligned} EU &= q\left(p\left(v - \theta x\right) + \left(1 - p\right)\left(v - \theta z\right)\right) \\ &\quad + \left(1 - q\right)\left(p\left(v - \theta x\right) + \left(1 - p\right)\left(v - \theta z\right)\right) \\ &= v - \theta\left[px + \left(1 - p\right)z\right]. \end{aligned}$$

But note that your choice of q is immaterial for the expected payoff. Hence any $q = p\epsilon(0, 1)$ would be a symmetric mixed strategy equilibrium. The reason that your choice of q is irrelevant is due to the fact that the with $v = -\gamma$ and $\delta = \theta$, $\alpha = \beta = 0$ own actions do only affect the payoff of the opponent.

Under continuous randomization (7) must hold, but it is immediate that any distribution fits. So any continuous mixed strategy with distribution $F(x)$ on $\mathbb{R}^{+}$ is an equilibrium.

5.5.3 Case III. $\eta \neq 0$

This case is further subdivided into the case where $V = 0$ and $V > 0$. But regarding the pure strategy equilibria, if present, these are unique by Proposition 2 as $\eta \neq 0$.

Subcase III1. $V = 0$ In case $V = 0$ and $\beta = 0$ we know by Propostion 2 part (b) that $(0, 0)$ is the unique pure strategy equilibrium. Furthermore, the differential equation (7) implies that with $\beta = 0$, there is no mixed strategy solution.

In case $V = 0$ and $\beta < 0$, there can be no equilibrium as both agents want to become the winner and profit from calling the highest possible number.

If $\beta > 0$, again by Proposition 2 part (a) $(0, 0)$ is the unique pure strategy equilibrium. But a mixed strategy equilibrium exists as well if $\eta > 0$ (there is no mixed strategy solution if $\eta < 0$). Either from solving $f/F = \beta/(\eta w)$ or directly from (17), the mixed strategy solution on $[0, D^{-\eta/\beta}]$ is

$$F(w) = Dw^{\beta/\eta} \tag{18}$$

with density

$$F(w) = D\frac{\beta}{\eta}w^{\beta/\eta-1}.$$

Note that

$$E[W] = \frac{\beta}{\beta+\eta}D^{-\eta/\beta}.$$

Clearly, it is necessary that $D > 0$ and $\beta/\eta > 0$. Since $\beta > 0$, it does not pay to raise the bid above $D^{-\eta/\beta}$, as this lowers the expected payoff at the upper bound. We conclude that there exists a continuum of mixed strategy equilibria, i.e. for any $D > 0$ (18) is a valid mixed strategy solution. Because the differential equation has the ratio form $f/F = \beta/(\eta w)$, note that there can be no mass point at zero. Only at the upper end would a mass point be possible, but this violates the requirement that the mass point is necessarily at zero by Proposition 2.

Example 13 *Suppose $v = 1$, $\gamma = -1$, so that $v+\gamma = 0$; and suppose that $\beta = \theta = 1$, $\delta = -1$, so that $\eta = 1$. Thus $W(x, y) = 1 - x + y$ and $L(x, y) = 1 - y$. Thus the contestants do not compete over the prize, but only over the externalities. Clearly $(0, 0)$ is the unique pure strategy equilibrium that yields both agents 1 for sure; if the opponent bids zero for sure, then raising one's bid to ε yields only $1 - \varepsilon$. The mixed strategy equilibrium $F(w) = w$ on $[0, 1]$ yields both agents $1/2$ in expectation since*

$$\begin{aligned} EU(x) &= \Pr\{\mathit{Win}\}(1 - x + E[y|y \le x]) + \Pr\{\mathit{Loose}\}(1 - E[y|y > x]) \\ &= x\left(1 - x + \frac{x}{2}\right) + (1 - x)\left(1 - \frac{1+x}{2}\right) = 1/2. \end{aligned}$$

Neither agent has an incentive to bid outside the support; for example given that the opponent plays $F(x) = x$, bidding $w = 1 + \varepsilon$, $\varepsilon > 0$ yields $1 - (1 + \varepsilon) + 1/2 < 1/2$. Neither can anyone gain by loading mass on a specific point in the support and this holds in particular for the pure strategy point zero. The pure strategy equilibrium though, Pareto dominates the mixed strategy solution.

Subcase III2. $V > 0$ If $V > 0$ and $\eta \neq 0$, then the differential equation (7) requires that $\beta \neq 0$ in any mixed strategy equilibrium (there exists a pure strategy equilibrium $V/(\delta - \theta)$ with $\beta = 0$ if $\eta < 0$), as otherwise the density $f(w) = 0$ for any $w > 0$.

There is no mixed strategy equilibrium if $V > 0$ and $\beta < 0$, as both agents want to become the winner and profit from calling the highest possible number plus the profit form the fact that $v > -\gamma$; but there can be no probability mass at infinity.

Thus necessarily $\beta > 0$. The solution in this case is (17)

$$F(w) = D\left(V + \eta w\right)^{\beta/\eta}, \; D > 0, \tag{19}$$

with density

$$F(w) = D\beta\left(V + \eta w\right)^{\beta/\eta - 1}.$$

We distinguish between the subcases with $\eta > 0$ and $\eta < 0$.

If $\eta > 0$, then solution (19) implies that there is a mass point as $F(0) = DV^{\beta/\eta} > 0$. But this contradicts Proposition 2 part (b), as $\eta < 0$ is a necessary requirement for a mass point if $V > 0$.

If $\eta < 0$, then Proposition 2 part (b) says that if there is a mass point, it should be at $-V/\eta$. However, this cannot be the case as $F(-V/\eta)$ from (19) would be unbounded. In fact note that at the upper bound u, where $F(u) = 1$,

$$u = \frac{1}{\eta} D^{-\eta/\beta} - \frac{V}{\eta} < -\frac{V}{\eta}.$$

But for any value $m\epsilon[0, u]$, $F(m) > 0$, implying there is a mass point at m, while if such a mass point existed it should be at $-V/\eta$. Again a contradiction.