A joint initiative of Ludwig-Maximilians University's Center for Economic Studies and the Ifo Institute for Economic Research

CESifo Area Conference on

Energy and Climate Economics

15 – 16 October 2010 CESifo Conference Centre, Munich

Intertemporal Risk Aversion, Stationarity and Discounting

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November 2007

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Abstract: The paper develops an axiomatic framework that derives a new relation between discounting and uncertainty evaluation. The von Neumann and Morgenstern axioms give rise to a richer form of risk attitude than captured in the discounted expected utility standard model. I derive three models that permit a more comprehensive uncertainty evaluation. These preference representations differ in the consistency requirements imposed on the evaluation of uncertain scenarios. A central result is that for an intertemporal risk averse decision maker a stationary risk evaluation can restrict the rate of pure time preference to zero (no impatience). Such a decision maker still gives reduced weight to expected future utility when uncertainty is increasing over time. If uncertainty is endogenous to the decision process, this new rationale for discounting can yield very different policy implications.

JEL Codes: D01, D60, D81, H43, Q01, Q51, Q54

Keywords: uncertainty, expected utility, recursive utility, risk aversion, intertemporal substitutability, stationarity, certainty additivity, temporal lotteries, intertemporal risk aversion, temporal resolution of risk, time preference, discounting, discount rate

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1 Introduction

The paper addresses limitations of the standard model in characterizing intertemporal trade-offs. It pays particular attention to uncertainty over future outcomes and to time preference. These two inputs are considered pivotal for long-term modeling and are currently the most hotly debated economic determinants of climate change evaluation (Stern 2007, Nordhaus 2007, Weitzman 2007, Arrow 2007).

The most widespread intertemporal evaluation framework is the discounted expected utility model. It contains, however, an implicit assumption of risk neutrality with respect to utility gains and losses. This form of risk neutrality is not implied by standard axioms, in particular not by those of von Neumann & Morgenstern (1944). The axiomatic approach taken in this paper shows that standard assumptions rationalize a significantly higher willingness to undergo preventive effort in order to prevent potential future damage than captured by the standard model. I derive three representations that differ in the consistency requirements imposed on the evaluation of uncertainty over time.

The assumption of preference stationarity permits to define *the pure rate of time preference*. Pure time preference characterizes the weight given to future utility (the 'utility discount factor'). A positive rate of pure time preference implies that future utility counts intrinsically less than present utility. I show that in the more comprehensive setting derived in this paper a positive rate of pure time preference stands in strong tension to a collection of rationality constraints. Instead, a decision maker exhibiting risk aversion with respect to utility gains and losses has a different rationale for discounting the future. Whenever uncertainty increases over time, his weight on future expected utility decreases over time, even it the rate of pure time preference is zero. The effect can resemble standard discounting – as well as non-constant discounting – on the basis of time preference. However, when addressing long-term evaluation problems where uncertainty is endogenous, like in the case of climate change, the two discounting rationales can have very different implications.

My setup closely relates to Kreps & Porteus (1978). In difference to their representation, I draw upon a result derived in Traeger (2007*b*) that permits to keep intertemporal aggregation additive by introducing nonlinear uncertainty aggregation. In the same paper I introduce the concept of *intertemporal risk aversion*. In a representation where aggregation of per period utility is additive over time, intertemporal risk aversion can be interpreted as risk aversion with respect to utility gains and losses. A particular feature of intertemporal risk aversion is that the respective risk measures stay one dimensional in the multi-commodity setting. They are independent of the commodity and its measure scale.[1] In a one commodity setting, the concept of intertemporal risk aversion closely relates to the disentanglement of Arrow Pratt risk aversion from intertemporal substitutability as introduced by Epstein & Zin (1989) and Weil (1990). The standard model implicitly assumes that the Arrow-Pratt measure of relative risk aversion coincides with the (inverse of the) elasticity of intertemporal substitution. Epstein & Zin (1991) give empirical evidence on asset pricing challenging this assumption. I show that a coinciding choice of Arrow Pratt risk aversion and aversion with respect to intertemporal substitution is by itself an assumption of (intertemporal) risk neutrality (see also Traeger 2007*b*).

The original axiomatization of *stationary* decision making goes back to Koopmans (1960) and expresses the idea that the mere passage of time does not change preferences. Koopmans' formulation and its successors crucially depend on the assumption of an infinite planning horizon and imply a strictly positive rate of pure time preference. I offer an alternative axiomatization for a finite planning horizon which, taken on its own, does not restrict time preference in any way. I distinguish between assuming a stationary evaluation of certain scenarios and of uncertain scenarios. In the standard discounted expected utility model these two assumptions have identical consequences, which is explained by the implicit adoption of intertemporal risk neutrality. For decision makers with a non-trivial intertemporal risk attitude, I show that the assumption of risk stationarity has stronger implications.

The general recursive evaluation models in this paper employ Kreps & Porteus' (1978) temporal lotteries. These temporal lotteries are recursive descriptions of uncertainty that generalize the standard characterization through probability distributions over consumption paths. The authors show that temporal lotteries permit an intrinsic *preference for early or late resolution of uncertainty.* This timing preference is intrinsic in the sense it holds even if the information obtained from uncertainty resolution cannot be used to refine decisions and improve outcomes or outcome probabilities. From a normative view on rational decision making it can be argued that a preference for early or late resolution

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[1] For consumption characteristics without a natural measure scale, such as quality or appearance, the fact whether an individual is risk averse or risk loving in the sense of the Arrow Pratt measure is up to the choice of measure scale.

of uncertainty should be restricted to situations where the information is instrumental to changing decisions and outcomes.[2] An according indifference assumption with respect to the timing of uncertainty resolution brings the resulting evaluation framework back to the standard description of uncertainty by means of probability measures over consumption paths and permits a non-recursive scenario evaluation. A combination of timing indifference and risk stationarity restricts the pure rate of time preference to zero and, thus, the utility discount factor to unity.

The assumption of stationarity closely relates to the concept of sustainability. Applied to intergenerational models, it captures that future generations depend on consumption and other welfare characteristics in a similar way as present generations. Intertemporal risk aversion captures a higher willingness to undergo preventive action in order to prevent a threat of harm in the future. Thus, it formalizes in a rigorous and consistent way a concern raised by the advocates of the precautionary principle which is often invoked in the sustainability debate. In this context, the paper shows that these two 'sustainability assumptions', in combination with widely accepted rationality constraints, imply that future generations' welfare should receive the same attention as that of current generations. Such an absence of pure time preference is often argued for in the sustainability literature and has a long history of prominent advocates including Ramsey (1928), Pigou (1932), Harrod (1948), Koopmans (1963), Solow (1974), and Broome (1992). However, so far the invoked arguments have only been based on ethical grounds and not on rationality constraints for intertemporal risk evaluation.

The paper is structured as follows. Section 2 introduces setup, axiomatic background and the concept of uncertainty aggregation rules. Section 3 discusses certainty stationarity. It motivates the finite time horizon version of the axiom, explains its relation to the standard form, and derives the corresponding preference representation. Section 4 introduces the concept of intertemporal risk aversion. It gives a simplified axiomatization that is adapted to the stationary setup and defines measures of intertemporal risk aversion. Section 5 elaborates the consequences of a stationary evaluation of uncertain scenarios. Section 6 analyzes the consequences of indifference to the timing of uncertainty resolution for certainty stationary preferences. Finally, section 7 brings together risk stationarity

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[2] It is widely believed that non-indifference to the timing of risk resolution is necessary to disentangle Arrow Pratt risk aversion from intertemporal substitutability. However, as I show in Traeger (2007*a*) that is not the case. The same is true for modeling correlation aversion.

and indifference to the timing of risk resolution and discusses the consequences for the weight given to future welfare. Section 8 concludes. Proofs are gathered in the appendix.

2 Preliminaries

Section 2 lays out the setup and the common axiomatic background for the representations in this paper. It also introduces nonlinear uncertainty aggregation rules that allow to capture intertemporal risk aversion without abandoning a time-additive evaluation.

2.1 Setup

Let X be a connected compact metric space that characterizes welfare determining factors within a period. These can be consumption levels as well as more abstract descriptions of quality or the state of an ecosystem. I refer to the elements x of X as outcomes. Time is discrete with planning horizon $T \in \mathbb{N}$. The space $\mathsf{X}^t = X^{T-t+1}$ denotes the Cartesian product equipped with the product metric. It characterizes the set of all certain consumption paths from period t to period T.[3] A consumption paths $\mathsf{x} \in \mathsf{X}^t$ is written $\mathsf{x} = (\mathsf{x}_t, \mathsf{x}_{t+1}, ..., \mathsf{x}_T) = (x_t, x_{t-1}, ..., x_T)$. Given $\mathsf{x} \in \mathsf{X}^t$, I define $(\mathsf{x}_{-i}, x) = (\mathsf{x}_t, ..., \mathsf{x}_{i-1}, x, \mathsf{x}_{i+1}, ..., \mathsf{x}_T) \in \mathsf{X}^t$ as the consumption path that coincides with x in all but the i^{th} period, in which it yields outcome x.

Uncertain outcomes are modeled as temporal lotteries. For any compact metric space Y, let $\Delta(Y)$ denote the set of Borel probability measures equipped with the Prohorov metric (giving rise to the topology of weak convergence).[4] Define $\tilde{X}_T = X$ and recursively $\tilde{X}_{t-1} = X \times \Delta(\tilde{X}_t)$ for all $t \in \{2, ..., T\}$, where each $\tilde{X}_t$ is equipped with the product metric. I denote $P_t = \Delta(\tilde{X}_t)$ and refer to the elements $p_t \in P_t$ as (period t) lotteries. Observe that in every period the decision maker has a *probability distribution over* the *outcome* in the respective period *and* the *probability distribution* over the future faced

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[3] I do not distinguish different sets of outcomes for different periods. X stands for the union of all possible outcomes perceivable in any period.

[4] For the interpretation of this paper I suggest an epistemic foundation of probabilities as found for example in Cox (1946,1961) or Jaynes (2003). Here probabilistic beliefs replace the notion of a binary logic. Lotteries do not (only) describe draws from an urn, but correspond to general characterizations of uncertainty with respect to possible outcomes.

in the next period. The set of degenerate lotteries in P_t is identified with the set $\tilde{X}_t$ of (generalized or multiperiod) outcomes in the usual way. A lottery yielding outcome $\tilde{x}_t$ with probability $p_t(\tilde{x}_t) = \lambda$ and outcome $\tilde{x}'_t$ with probability $p_t(\tilde{x}'_t) = 1 - \lambda$ is written as $\lambda\tilde{x}_t + (1-\lambda)\tilde{x}'_t \in P_t$. A 'plus' sign between outcomes always characterizes a lottery.[5] Preferences in period t are defined on the set P_t and denoted by $\succeq_t$.[6] For continuous functions $u : X \to \mathbb{R}$ and $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$, evaluating outcomes x respectively $\tilde{x}_t$, I denote $U = [\underline{U}, \overline{U}] = \text{range}(u)$ and $\tilde{U}_t = \text{range}(\tilde{u}_t)$. For any metric space Y I denote by $\mathcal{C}^0(Y)$ the set of continuous functions from Y into the reals. Finally, the group of strictly positive affine transformations is denoted $\mathsf{A} = \{\mathsf{a} : \mathbb{R} \to \mathbb{R} : \mathsf{a}(z) = a\,z + b\,,\ a, b \in \mathbb{R}, a > 0\}$ with elements $\mathsf{a} \in \mathsf{A}$.

2.2 Axiomatic Background

The paper assumes throughout that the von Neumann Morgenstern axioms, additive separability on certain consumption paths, and time consistency are satisfied. I briefly review these axioms below.

The first three axioms are close relatives to the ones suggested by von Neumann & Morgenstern (1944) in an atemporal framework for choice under uncertainty.

A1 (weak order) For all $t \in \{1, ..., T\}$ $\succeq_t$ is transitive and complete, i.e.:

- transitive: $\forall\, p, p', p'' \in P_t :\ p \succeq_t p'$ and $p' \succeq_t p'' \Rightarrow p \succeq_t p''$.
- complete: $\forall\, p, p' \in P_t : p \succeq_t p'$ or $p' \succeq_t p$.

Axiom A1 assumes that the decision maker can rank all lotteries (completeness). Moreover, if one lottery is preferred to a second and the second is preferred to a third, then the first lottery should also be preferred to the third (transitivity).[7]

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[5] As X and $\tilde{X}_t$ are only assumed to be compact metric there is no immediate addition defined for their elements. In case the spaces are additionally equipped with a vector space or field structure, the vector composition will not coincide with the "+" used here. The "+" sign used here alludes to the additivity of probabilities.

[6] The relations $\succeq_t$ are required to be reflexive (axiom A1). The asymmetric part is denoted by $\succ_t$ and interpreted as strict preference. The symmetric part of the relation $\succeq_t$ is denoted by $\sim_t$ and interpreted as indifference. Non-indifference is denoted by $\not\sim_t$ and defined as $\not\sim_t \equiv P_t \times P_t \setminus \sim_t$.

[7] Note that, within a normative context, axiom A1 should be interpreted as "if a decision maker had

A2 (independence) For all $t \in \{1, ..., T\}$ and for all $p, p', p'' \in P_t$:

$$p \sim_t p' \quad \Rightarrow \quad \lambda\, p + (1-\lambda)\, p'' \sim_t \lambda\, p' + (1-\lambda)\, p'' \quad \forall\, \lambda \in [0,1] \ .$$

The independence axiom states the following. Let a decision maker be indifferent between a lottery p and another lottery p'. Offer him two compound lotteries, both starting out with the toss of a biased coin (tail comes up with probability λ). In both lotteries the decision maker enters the same third lottery p'' if head comes up. However, if tail comes up, the decision maker faces lottery p in the first compound lottery and the lottery p' in the second. Recalling that the decision maker is indifferent between lotteries p and p', the independence axiom requires the decision maker to be indifferent between the two compound lotteries as well.[8]

A3 (continuity) For all $t \in \{1, ..., T\}$ and for all $p \in P_t$:

The sets $\{p' \in P_t : p' \succeq_t p\}$ and $\{p' \in P_t : p \succeq_t p'\}$ are closed in P_t.

Continuity A3 assures that infinitesimally small changes in the probabilities do not result in finitely large changes in the evaluation. In particular, continuity implies the slightly weaker Archimedian axiom used by von Neumann & Morgenstern (1944).

In order to match the widespread model of additively separable utility over time on certain consumption paths, I introduce the following axiom taken from Wakker (1988).

A4 (certainty separability)

i) For all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^1$, $x, x' \in X$ and for all $t \in \{1, ..., T\}$:

$$(\mathsf{x}_{-t}, x) \succeq_1 (\mathsf{x}'_{-t}, x) \ \Leftrightarrow \ (\mathsf{x}_{-t}, x') \succeq_1 (\mathsf{x}'_{-t}, x') \, .$$

ii) If $T = 2$ additionaly: For all $x_t, x'_t, x''_t \in X$, $t \in \{1, 2\}$

$$(x_1, x_2) \sim_1 (x'_1, x''_2) \ \wedge \ (x'_1, x'_2) \sim_1 (x''_1, x_2) \ \Rightarrow \ (x_1, x'_2) \sim_1 (x''_1, x''_2) \, .$$

Wakker (1988) calls part *i*) of the axiom coordinate independence. It requires that the

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the capacities to rank all possible outcomes, then his ranking should satisfy transitivity" rather than as an assumption that the decision maker actually ranks all possible outcomes.

[8] For a discussion of descriptive limitations of the independence axiom see e.g. Starmer (2000). Note that the independence axiom can be replaced by a collection of much weaker axioms elaborated by Chew & Epstein (1989, 110) in the case of indifference to the timing of risk resolution, a setting treated in sections 6 and 7.

choice between two consumption paths does not depend on period t consumption, whenever the latter coincides for both paths. Part *ii*) is known as the Thomsen condition. It is required only if the model is limited to $T = 2$ periods.[9] Axiom 4 is the main ingredient to allow for a certainty additive representation of the form $\sum_{t=1}^{T} u_t(\mathrm{x}_t)$. So far the axioms allow the representing utility functions u_t to change arbitrarily over time. That characteristic is tamed by the stationarity assumptions introduced in this paper.

Preferences in different periods are connected by the following consistency axiom adapted from Kreps & Porteus (1978).

A5 (time consistency) For all $t \in \{1, ..., T-1\}$:

$$(x_t, p_{t+1}) \succeq_t (x_t, p'_{t+1}) \quad \Leftrightarrow \quad p_{t+1} \succeq_{t+1} p'_{t+1} \quad \forall\, x_t \in X,\ p_{t+1}, p'_{t+1} \in P_{t+1} \ .$$

The axiom requires that a decision maker who prefers in period t a certain outcome followed by lottery p_{t+1} rather than p'_{t+1}, also prefers p_{t+1} in period $t+1$.

Finally, for convenience in presentation, I assume that there exist at least two non-indifferent consumption paths in the decision maker's choice set.

A0 (nondegeneracy) There exist $\mathrm{x}, \mathrm{x}' \in \mathrm{X}^1$ such that $\mathrm{x} \not\sim_1 \mathrm{x}'$.

2.3 Uncertainty Aggregation Rules

The preference representations in this paper rely on non linear aggregation of utility over uncertainty. For any $t \in \{1, ..., T\}$ and strictly monotonic function $f_t \in \mathcal{C}^0(\mathbb{R})$ I define an *uncertainty aggregation rule* as the functional $\mathcal{M}^{f_t} : \Delta(\tilde{X}_t) \times \mathcal{C}^0(\tilde{X}_t) \to \mathbb{R}$ with[10]

$$\mathcal{M}^{f_t}(p_t, \tilde{u}_t) = f_t^{-1} \int_{\tilde{X}_t} dp_t \, f_t \circ \tilde{u}_t \ .$$

The uncertainty aggregation rule takes as input the decision maker's perception of uncertainty, expressed by a probability measure p_t on $\tilde{X}_t$, and a valuation of the degenerate outcomes expressed by a real valued function $\tilde{u}_t$ on $\tilde{X}_t$. The uncertainty aggregation rule

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[9] In the case of two periods parts *i*) and *ii*) can also be replaced by the single requirement of triple cancellation (see Wakker 1988, 427). Moreover, part *ii*) can be replaced by a slightly weaker but slightly more involved assumption known as the hexagon condition (Wakker 1989, 47 et sqq., 67).

[10] By continuity of $f_t \circ \tilde{u}_t$ and compactness of $\tilde{X}_t$, Lesbeque's dominated convergence theorem ensures integrability (Billingsley 1995, 209).

weighs these utility values with the function f_t, aggregates them, and applies the inverse of f_t to renormalize the resulting expression. For degenerate outcomes an uncertainty aggregation rule returns the value of $\tilde{u}_t$ itself, i.e. $\mathcal{M}^{f_t}(\tilde{x}_t, \tilde{u}_t) = \tilde{u}_t(\tilde{x}_t)$.

The simplest example of an uncertainty aggregation rule is the expected value operator which is obtained for $f_t = \text{id}$. An uncertainty aggregation rule that will be generated by different axioms in this paper corresponds to the parametrization $f_t(z) = \exp(\xi z) = \big(\exp(z)\big)^\xi$ yielding:

$$\mathcal{M}^{\exp^\xi}(p_t, \tilde{u}_t) = \frac{1}{\xi} \ln \left[\int_{\tilde{X}_t} dp_t \exp(\xi \tilde{u}_t) \right] .$$

It is defined for $\xi \in \mathbb{R}$ with $\mathcal{M}^{\exp^0}(p_t, \tilde{u}_t) \equiv \lim_{\xi \to 0} \mathcal{M}^{\exp^\xi}(p_t, \tilde{u}_t) = E_{p_t} \tilde{u}_t$ (see proof of theorem 3). In the limit of ξ going to plus or minus infinity, the uncertainty aggregation rule $\mathcal{M}^{\exp^\xi}$ only considers the extreme outcomes (abandoning continuity in the probabilities): $\lim_{\xi \to \infty} \mathcal{M}^{\exp^\xi}(p_t, \tilde{u}_t) = \max_{\tilde{x}_t} \tilde{u}_t(\tilde{x}_t)$ and $\lim_{\xi \to -\infty} \mathcal{M}^{\exp^\xi}(p_t, \tilde{u}_t) = \min_{\tilde{x}_t} \tilde{u}_t(\tilde{x}_t)$. In general it can be shown that the smaller is ξ, the lower is the value (or certainty equivalent utility) that the respective uncertainty aggregation rule brings about.[11]

3 Certainty Stationarity

The section introduces stationarity for settings with a finite planning horizon. First, I introduce the axiomatic characterization. Then, I give a preference representation that admits a more general evaluation of uncertainty than the standard model.

3.1 Axiomatic Description for a Finite Planning Horizon

An almost ubiquitous assumption in economic modeling is that utility evaluation of consumption coincides in different periods (discounted utility model). While the assumption is usually adopted independent of an agent's planning horizon, the assumption has been been provided with an axiomatic foundation only in the case of an infinite planning horizon. In the latter setting, Koopmans (1960) has shown that the pure rate of time preference has to be constant and strictly positive.

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[11] The uncertainty aggregation rule $\mathcal{M}^{f_t}$ is induced by the so called quasi-arithmetic or generalized mean in a way made precise in Traeger (2007*b*).

Koopmans' characterization of stationarity requires that a decision maker prefers a consumption path x over another consumption path x' in the present, if and only if, he prefers a consumption path (x^0, x) over a consumption path (x^0, x') in the present (Koopmans 1960, see page 10 below for details). Such an axiomatization uses the fact that for an infinite time horizon, adding an additional outcome does not change the length of a consumption path. Both paths, x and (x^0, x), are elements of X^∞ and can be compared by the same preference relation. On the contrary, with a finite planning horizon, the paths x and (x^0, x) differ in length and, thus, cannot be compared by means of the same preference relation $\succeq$. The following axiom is applicable in the setting with a finite planning horizon and, there, yields the standard discounted utility model for the evaluation of certain consumption paths.

A6 (certainty stationarity) For all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^2$ and all $x \in X$ holds

$$(\mathsf{x}, x) \succeq_1 (\mathsf{x}', x) \quad \Leftrightarrow \quad \mathsf{x} \succeq_2 \mathsf{x}' \,. \tag{1}$$

On the right hand side of the equivalence, the decision maker faces a comparison between x and x' as consumption paths starting in period 2. On the left hand side of the equivalence, the decision maker faces a comparison between x and x' as consumption paths starting in period 1. The additional outcome x, which is commonly added to the paths x and x', makes (x, x) and (x', x) choice objects of the appropriate length, so that they can be compared in period 1 by the preference relation $\succeq_1$ of a decision maker with time horizon T. The important property of the axiom is that the decision maker's preference over the (certain) consumption paths is independent of their starting point.[12]

For the best interpretation of axiom 6 it is worthwhile adopting a rolling time horizon and splitting the equivalence (1) into two separate statements. Assume that a decision maker in period 1, planning with time horizon T, prefers consumption plan (x, x) over plan (x', x). Confront him in period 2 with the exact same consumption paths (x, x) and (x', x). For this purpose let the decision maker plan ahead in period 2 the same number of periods as he does in period 1, implying the new time horizon $T+1$. Formally, I denote these preferences of the decision maker in period 2 with time horizon $T+1$ by $\succeq_{2|T+1}$. I

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[12] Note the difference to time consistency. The latter is a condition on two consumption paths starting in the same period, which yield a common outcome in the first period. Then, the *continuation* of the paths in the next period should be ranked the same way as the complete paths in the earlier period.

assume that the decision maker ranks (or plans to rank) the consumption paths in both choice situations the same way. Requiring the latter for all consumption paths yields the condition

$$(\mathsf{x}, x) \succeq_{1|T} (\mathsf{x}', x) \quad \Leftrightarrow \quad (\mathsf{x}, x) \succeq_{2|T+1} (\mathsf{x}', x) \tag{2}$$

for all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^2$ and $x \in X$. Condition (2) most clearly captures the intuition of stationarity in the sense that the mere passage of time does not change the evaluation.

However, so far assumption (2) does not imply any restriction on the set of preferences. $(\succeq_t)_{t\in\{1,\dots,T\}} = (\succeq_{t|T})_{t\in\{1,\dots,T\}}$. The second step in the reasoning relates the preference relation $\succeq_{2|T+1}$ to the relation $\succeq_2 = \succeq_{2|T}$. Both preference relations specify how the decision maker evaluates (anticipates evaluating) consumption plans over the future in period 2. The relation $\succeq_{2|T}$ specifies his ranking when he plans ahead until period T, and the relation $\succeq_{2|T+1}$ states his ranking when he plans ahead until period $T+1$. Accepting stationarity in the rolling time horizon sense of equation (2), axiom A6 requires the following relation between $\succeq_{\cdot|T}$ and $\succeq_{\cdot|T+1}$:

$$\mathsf{x} \succeq_{2|T} \mathsf{x}' \quad \Leftrightarrow \quad (\mathsf{x}, x) \succeq_{2|T+1} (\mathsf{x}', x) \tag{3}$$

for all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^2$ and $x \in X$. If two scenarios are evaluated with a time horizon of $T+1$ and yield the same outcome in period $T+1$, then, an evaluation based only on a time horizon T yields the same ranking of the scenarios. The assumption is a weak form of future independence.

3.2 Relation to the Infinite Planning Horizon Setting

The paragraph relates the axiomatic reasoning above to the situation where agents have an infinite planning horizon. Denote the consumption paths corresponding to (x, x) and (x', x) by $\mathsf{x}^\infty, \mathsf{x}'^\infty \in X^\infty$. By *time consistency* the right hand side of equation (2) is equivalent to $(x, \mathsf{x}^\infty) \succeq_{1|\infty+1} (x, \mathsf{x}'^\infty)$ for all $\mathsf{x}^\infty, \mathsf{x}'^\infty \in X^\infty$ and $x \in X$. But, in the infinite horizon setting it holds $\succeq_{1|T+1} = \succeq_{1|\infty+1} = \succeq_{1|\infty} = \succeq_{1|T}$, a relation that makes equation (3) dispensable. Together, I arrive at the standard axiom of stationarity for the infinite planning horizon:

$$\mathsf{x}^\infty \succeq_{1|\infty} \mathsf{x}'^\infty \Leftrightarrow (x, \mathsf{x}^\infty) \succeq_{1|\infty} (x, \mathsf{x}'^\infty) \tag{4}$$

for all $x \in X$ and all $\mathsf{x}^\infty, \mathsf{x}'^\infty \in X^\infty$, dating back to Koopmans (1960, 294)[13]. Hence, at first sight, the second assumption corresponding to equation (3) does not seem necessary with an infinite planning horizon. However, this conjecture is not correct. In order to satisfy equation (4), i.e. equation (2) with an infinite time horizon, in a standard representation, a positive rate of pure time preference turns out necessary. Therefore, the weight given to future consumption converges to zero and the coinciding outcomes in the 'last' period of the planning horizon are insignificant to the ranking. Thus, with an infinite planning horizon and the induced intrinsic devaluation of the future equation (2) already implies equation (3).

3.3 The Certainty Stationary Representation

The following representation holds for preferences satisfying axioms A1-A5 and a stationary evaluation of certain consumption paths in the sense of axiom A6.

Theorem 1: Let a sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfy axiom A0. The sequence $\succeq$ satisfies axioms A1-A5 and certainty stationarity A6, if and only if, there exist a continuous function $u : X \to \mathbb{R}$, a discount factor $\beta \in \mathbb{R}_{++}$, and strictly increasing and continuous functions $f_t : \mathbb{R} \to \mathbb{R}$ for all periods $t \in \{1, ..., T\}$, such that defining the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by $\tilde{u}_T(x_T) = u(x_T)$ and recursively

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = u(x_{t-1}) + \beta\, \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \tag{5}$$

the representing equivalence

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t \tag{6}$$

holds for all periods $t \in \{1, ..., T\}$.

Moreover, the functions $u, f_1, ..., f_T$ and $u', f'_1, ..., f'_T$ both represent $\succeq$ in the above sense, if and only if, there exist positive affine transformations $\mathsf{a}_0, \mathsf{a}_1, ..., \mathsf{a}_T \in \mathsf{A}$ such that $u' = \mathsf{a}_0 u$ and $f'_t|_{\tilde{U}'_t} = \mathsf{a}_t f_t|_{\tilde{U}_t}\, {\mathsf{a}_0}^{-1}$.

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[13] Koopmans (1960) formulates his postulates in terms of utility functionals. However, the translation of his postulate 4 into the preference setup is immediate. His general axiomatic setting is translated into preferences in Koopmans (1972), again with stationarity corresponding to postulate 4.

The representation theorem recursively constructs an aggregate utility function $\tilde{u}_t$ that depends on the utility in the respective period $u(x_t)$ and the utility derived from a particular lottery p_{t+1} over the future. While aggregation of current utility and future utility is additive in all periods, uncertainty evaluation relies on the generally nonlinear uncertainty aggregation rules $\mathcal{M}^{f_t}$. Graphically, the construction of aggregate period t utility recursively works its way through the decision tree corresponding to the lottery p_t. Starting in the last period, the representation evaluates possible outcomes by u and aggregates over last period subtrees employing the uncertainty aggregation rule $\mathcal{M}^{f_T}$. The second step evaluates possible outcomes in the preceding period, again by u, and adds the values to the discounted utility of the attached subtree (yielding the $\tilde{u}_{T-1}$ values). Uncertainy evaluation and time aggregation are alternately repeated until the aggregate utility for period t is obtained. A non-recursive evaluation of the decision trees is only possible for special cases which are discussed in sections 6 and 7.

Recalling that for degenerate lotteries it holds $\mathcal{M}^{f_t}(\tilde{x}_t, \tilde{u}_t) = \tilde{u}_t(\tilde{x}_t)$, the evaluation of certain consumption paths simply becomes

$$\mathrm{x}^t \succeq_t \mathrm{x}'^t \quad \Leftrightarrow \quad \sum_{\tau=t}^{T} \beta^{\tau-t}\, u(\mathrm{x}^t_\tau) \geq \sum_{\tau=t}^{T} \beta^{\tau-t}\, u(\mathrm{x}'^t_\tau)\,. \tag{7}$$

Thus, for certain consumption paths, the model coincides with the standard discounted utility framework. Observe that the discount factor β in theorem 1 can be smaller, greater or equal to unity. For uncertain consumption, however, theorem 1 gives a more general evaluation framework that avoids an implicit assumptions of risk neutrality present in the intertemporally additive expected utility standard model. This risk neutrality assumption is discussed in the next section.

4 Intertemporal Risk Aversion

In Traeger (2007*b*) I have introduced the concept of intertemporal risk attitude in a general framework with non-stationary preferences. This section gives a slightly simplified definition for decision makers with stationary preferences. The concept of intertemporal risk attitude yields a one dimensional measure of risk aversion for a multi-commodity world. In difference to the Arrow Pratt measure of risk aversion, it is independent of the measure scale used for the goods under observation. I first give an axiomatic characterization and, then, characterize intertemporal risk attitude in the representation of theorem 1.

4.1 Axiomatic Characterization of Intertemporal Risk Attitude

This section employs a special notation for constant consumption paths. The path $\bar{\mathsf{x}}^t = (\bar{x}, \bar{x}, ..., \bar{x})$ denotes the certain constant consumption path that gives consumption $\bar{x}$ from t until T. Moreover, recall that for $\mathsf{x} \in \mathsf{X}^t$ the consumption path $(\bar{\mathsf{x}}_{-i}, \mathsf{x}_i)$ denotes the consumption path where the i^{th} entry of $\bar{\mathsf{x}}$ is replaced by outcome x_i. Then, the lottery $\sum_{i=t}^{T} \frac{1}{T-t+1} \, (\bar{\mathsf{x}}_{-i}, \mathsf{x}_i)$ delivers consumption paths with outcomes $\bar{x}$ in all but one period. The lottery draws with equal probability the period i in which the outcome $\bar{x}$ is replaced by outcome x_i.

A decision maker exhibits *weak intertemporal risk aversion* in period $t < T$, if and only if, the following axiom is satisfied:

A7$^{\mathrm{w}}$ (weak intertemporal risk aversion) For all $\bar{\mathsf{x}}, \mathsf{x} \in \mathsf{X}^t$ holds

$$\bar{\mathsf{x}} \sim_t \mathsf{x} \quad \Rightarrow \quad \bar{\mathsf{x}} \;\succeq_t\; \sum_{i=t}^{T} \frac{1}{T-t+1} \, (\bar{\mathsf{x}}_{-i}, \mathsf{x}_i).$$

A decision maker exhibits *strict intertemporal risk aversion* in period $t < T$, if and only if, the following axiom is satisfied:

A7$^{\mathrm{s}}$ (strict intertemporal risk aversion) For all $\bar{\mathsf{x}}, \mathsf{x} \in \mathsf{X}^t$ holds

$$\bar{\mathsf{x}} \sim_t \mathsf{x} \quad \wedge \quad \exists\, \tau \in \{t, ..., T\} \text{ s.th. } (\bar{\mathsf{x}}_{-\tau}, \mathsf{x}_\tau) \not\sim_t \bar{\mathsf{x}}$$

$$\Rightarrow \quad \bar{\mathsf{x}} \;\succ_t\; \sum_{i=t}^{T} \frac{1}{T-t+1} \, (\bar{\mathsf{x}}_{-i}, \mathsf{x}_i).$$

I start with the interpretation of the strict axiom. The first part of the premise in axiom A7$^{\mathrm{s}}$ states that a decision maker is indifferent between a constant consumption path delivering outcome $\bar{x}$ in every period and the consumption path x. The second part of the premise requires that at least some outcome x_τ is considered non-indifferent to $\bar{x}$ so that the path $(\bar{\mathsf{x}}_{-\tau}, \mathsf{x}_\tau)$ is non-indifferent to the path $\bar{\mathsf{x}}$. Together, the two assumptions in the premise imply that some of the reassembled consumption paths $(\bar{\mathsf{x}}_{-i}, \mathsf{x}_i)$ with $i \in \{t, ..., T\}$ are preferred to the constant path while others are judged inferior. The outcomes x_t that are better and those that are worse than $\bar{x}$ balance in a way that receiving all with certainty makes the decision maker indifferent with respect to the constant consumption path. The second line of axiom A7$^{\mathrm{s}}$ states that an intertemporal risk averse decision maker prefers the certain constant consumption path over the lottery that exchanges, with equal probability, one of the outcomes. For some draws of the period i the decision maker will receive a consumption path that makes him better off, but for others he receives a path that makes him worse off.

The interpretation of the *weak* axiom A7$^{\mathrm{w}}$ is analogous, only that the consumption path x is allowed to coincide with $\bar{\mathrm{x}}$, and the implication only requires that the lottery is not strictly preferred to the certain and constant consumption path. If axiom A7$^{\mathrm{s}}$ [A7$^{\mathrm{w}}$] is satisfied with $\succ_t$ [$\succeq_t$] replaced by $\prec_t$ [$\preceq_t$], the decision maker is called a strong [weak] *intertemporal risk seeker*. If a decision maker's preferences satisfy weak intertemporal risk aversion as well as weak intertemporal risk seeking, the decision maker is called *intertemporal risk neutral.*

In the following, I give an alternative interpretation of axiom A7$^{\mathrm{s}}$, relating it to the representation of theorem 1. Here, the first part of the premise requires that for two consumption paths $\bar{\mathrm{x}}$ and x the discounted per period utility adds up to the same aggregate utility (see equation 7). The second part of the premise requires that at least in one period the utility gained from consumption path x differs from the utility gained from the outcome $\bar{x}$. Then, the lottery in axiom A7$^{\mathrm{s}}$ delivers *in expectation* the same utility as the certain consumption path $\bar{\mathrm{x}}$. A decision maker is defined to be strictly intertemporal risk averse if he prefers the certain consumption path $\bar{\mathrm{x}}$ over the lottery that leaves him either worse or better off, and yields the same utility as the certain consumption path in expectation. Here, intertemporal risk aversion can be interpreted as risk aversion with respect to welfare gains and losses, where welfare is described by the certainty additive utility function.[14]

4.2 Functional Characterization of Intertemporal Risk Attitude

The following theorem characterizes intertemporal risk aversion in terms of the preference representation of theorem 1.

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[14] An alternative definition of intertemporal risk aversion that only involves a lottery over two consumption paths is given in Traeger (2007*b*)

Theorem 2: Let the functions $f_1, ..., f_T$ represent the set of preferences $\succeq = (\succeq_t)_{t \in \{1,...,T\}}$ in the sense of theorem 1. The following assertions holds for all $t \in \{1, ..., T-1\}$:
a) A decision maker is strictly intertemporal risk averse [seeking] in period t in the sense of axiom A7$^{\text{s}}$, if and only if, $f_t|_{\tilde{U}_t}$ is strictly concave [convex].
b) A decision maker is weakly intertemporal risk averse [seeking] in period t in the sense of axiom A7$^{\text{w}}$, if and only if, $f_t|_{\tilde{U}_t}$ is strictly concave [convex].
c) A decision maker is intertemporal risk neutral in period t, if and only if, $f_t|_{\tilde{U}_t}$ is linear.

Intertemporal risk attitude is described by the curvature of the functions f_t. As pointed out at the end of the preceding section, in the representation of theorem 1, intertemporal risk aversion can be interpreted as risk aversion with respect to utility gains and losses. Thus, the standard intuition associating concavity with risk aversion applies. In difference to measures of atemporal risk aversion, the argument of f_t is not measured in consumption but in utility units. Precisely, it is measured in 'current value' certainty additive utility units: Combining equations (5) and (6) shows that f_t weights utility $\tilde{u}_t(x_t, p_{t+1}) = u(x_t) + \beta \mathcal{M}^{f_t}(p_{t+1}, \tilde{u}_{t+1})$. Thus, it aggregates utility from period t consumption in undiscounted 'current value' units.

Together with theorem 1, part *c*) of theorem 2 gives an axiomatic characterization of the expected discounted utility standard model: If intertemporal risk neutrality is required in all periods,[15] then f_t is linear in all periods. A linear parametrization makes the uncertainty aggregation rule itself linear, i.e. $\mathcal{M}^{f_t} = \text{E}$. With linear uncertainty aggregation the evaluation of the lotteries no longer has to be carried out recursively. The decision tree can be 'collapsed' and the expected value operator in theorem 1 can be pulled through to the first period (see also section 6).

In order to derive a quantitative characterization of risk attitude, I define for a twice differentiable function f_t the measures of *relative intertemporal risk aversion* as the functions $\text{RIRA}^*_t : \tilde{U}_t \to \mathrm{I\!R}$ with

$$\text{RIRA}^*_t(z) = -\frac{\frac{d^2}{dz^2} f_t(z)}{\frac{d}{dz} f_t(z)} \, z \, ,$$

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[15] Instead of characterizing intertemporal risk neutrality by joint weak intertemporal risk aversion and seeking, it can obviously be defined by replacing $\succeq_t$ in axiom A7$^{\text{w}}$ by $\sim_t$.

and of *absolute intertemporal risk aversion* as the functions $\text{AIRA}^*_t : \tilde{U}_t \to \mathrm{I\!R}$ with

$$\text{AIRA}^*_t(z) = -\frac{\frac{d^2}{dz^2} f_t(z)}{\frac{d}{dz} f_t(z)} \,.$$

For the preference representation in theorem 1 with f_t as in the theorem and outcome $\tilde{x}_t$ I define $\text{RIRA}^*_t[\tilde{x}_t] = \text{RIRA}^*_t(z)|_{z=\tilde{u}_t(\tilde{x}_t)}$ and $\text{AIRA}^*_t[\tilde{x}_t] = \text{AIRA}^*_t(z)|_{z=\tilde{u}_t(\tilde{x}_t)}$ as the intertemporal risk aversion measures at the point in consumption space $\tilde{x}_t$. However, due to the affine freedom $\mathfrak{a}_0$ in the representation of theorem 1, these risk measures are not yet well defined. The reason is discussed in detail in Traeger (2007*b*), however, the following intuition is straight forward. The standard Arrow Pratt measures of relative and absolute risk aversion are not well defined unless a cardinal measure scale for the consumption commodity under observation is fixed. Similarly, the measures of absolute and relative intertemporal risk aversion are not well defined, unless a cardinal measure scale is fixed. Intertemporal risk aversion and the curvature of f_t measure risk aversion with respect to welfare gains and losses, where welfare is measured by the abstract concept of the intertemporally additive utility function u. By observing certain trade-offs, the preference relation fixes this measure scale up to affine transformations. Analogously to standard risk aversion measures, the measure of relative intertemporal risk aversion RIRA^*_t will be well defined only if also a zero welfare level is fixed. The measure of absolute intertemporal risk aversion AIRA^*_t will be well defined only if a unit of welfare is fixed.[16]

Proposition 1: Let a sequence of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ be represented in the sense of theorem 1.

a) Choose $x^0 \in X$ and fix $u(x^0) = 0$.

Then the risk measures RIRA^*_t in the representation of theorem 1 are well defined for $t \in \{1, ..., T\}$ and the value of $\text{RIRA}^*_t[\tilde{x}_t]$ only depends on the preferences $\succeq$ and the point $\tilde{x}_t$ in consumption space.

b) Choose $x^1, x^2 \in X$ satisfying $x^1 \succ_T x^2$ and fix $u(x^1) - u(x^2) = 1$.

Then the risk measures AIRA^*_t in the representation of theorem 1 are well defined

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[16] The affine transformation $\mathfrak{a}_0$ in the uniqueness part of theorem 1 points out what happens if unit and/or zero level are not fixed. For example a doubling of u would correspond to an increase in welfare range and a decrease in welfare unit. The function f_t measuring risk aversion with respect to welfare gains and losses has to offset this change in measurement by applying the inverse of $\mathfrak{a}_0$. While this adjustment is necessary to represent the same preferences, it changes the numerical value of the intertemporal risk measures.

for $t \in \{1, ..., T\}$ and the value of $\text{AIRA}^*_t[\tilde{x}_t]$ only depends on the preferences $\succeq$ and the point $\tilde{x}_t$ in consumption space.

Bring to mind that preferences of a decision maker are over real outcomes, not over numbers in the Euclidean space which only serve as a representation tool and depend on the chosen coordinate system (i.e. measure scale). As elaborated in Traeger (2007*b*), the proposition implies in particular that the risk measures RIRA^*_t and AIRA^*_t do not depend on the measure scale applied to measure the point $\tilde{x}_t$ in consumption space. This invariance is particularly attractive when modeling risk aversion with respect to characteristics such as quality which are generally not equipped with a natural measure scale. Then, whether a decision maker is risk loving of risk seeking in the sense of the Arrow Pratt measures is only a question of the applied measure scale (Traeger 2007*b*). On the contrary, the measures of intertemporal risk aversion are invariant to changes in the measure scale of goods and well defined and one dimensional also in a multi-commodity setting.

Fixing both, the zero welfare level in a) and the unit level in b) makes the function u in theorem 1 unique and eliminates the affine freedom a^0. If the zero level has already been fixed, unit choice can also be obtained by simply picking $x^1 \in X$ with $\mathsf{x}^1 \succ_T x^0$ and fixing $u(x^1) = 1$ (which corresponds in part b) to the choice $x^2 = x^0$). Instead of fixing both, unit and zero level, also the range of u can be fixed resulting in the same uniqueness result. As pointed out above, the functions f_t and, thus, RIRA^*_t and AIRA^*_t measure intertemporal risk aversion with respect to 'current value' welfare changes. In difference, the intertemporal risk aversion measures introduced in Traeger (2007*a*,*b*) for the general non-stationary setting measure intertemporal risk aversion with respect to 'present value' welfare changes. For this reason I have marked the 'current value' measures RIRA^*_t and AIRA^*_t introduced here for the stationary setting with an asterisk.

5 Risk Stationarity

In the certainty stationary setting of representation theorem 1, the functions f_t characterizing intertemporal risk aversion are allowed to vary arbitrarily over time. This section extends stationarity to the evaluation of uncertain consumption plans.

5.1 Axiomatic Description for a Finite Planning Horizon

The risk stationarity axiom extends the assumption that the passage of time does not change preferences to the ranking of uncertain consumption plans. In order to obtain such a stationary evaluation of general uncertain outcomes, it turns out sufficient to require stationarity only for 'coin toss' compositions of certain consumption paths, i.e. probability a half mixtures of type $\frac{1}{2}\mathsf{x} + \frac{1}{2}\mathsf{x}'$. As for certainty stationarity in section 3.1, I split the motivation of the axiom into two steps. The first requirement, corresponding to equation (2) becomes:

$$\tfrac{1}{2}(\mathsf{x}, x) + \tfrac{1}{2}(\mathsf{x}', x) \succeq_{t|T} (\mathsf{x}'', x) \;\;\Leftrightarrow\;\; \tfrac{1}{2}(\mathsf{x}, x) + \tfrac{1}{2}(\mathsf{x}', x) \succeq_{t+1|T+1} (\mathsf{x}'', x) \tag{8}$$

for all $\mathsf{x}, \mathsf{x}', \mathsf{x}'' \in \mathsf{X}^{t+1}$, $x \in X$ and $t \in \{1, ..., T\}$. With a rolling time horizon, the mere passage of time does not change the ranking between different uncertain scenarios.[17]

The second step to arrive at the axiom of risk stationarity connects the relations $\succeq_{t+1|T}$ and $\succeq_{t+1|T+1}$. Again, I require that scenarios whose outcomes coincide in the last period of a finite planning horizon $T+1$ are ranked the same way when applying only a planning horizon of T. This statement translates into the equivalence

$$\tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}' \succeq_{t+1|T} \mathsf{x}'' \quad\Leftrightarrow\quad \tfrac{1}{2}(\mathsf{x}, x) + \tfrac{1}{2}(\mathsf{x}', x) \succeq_{t+1|T+1} (\mathsf{x}'', x) \tag{9}$$

for all $\mathsf{x}, \mathsf{x}', \mathsf{x}'' \in \mathsf{X}^{t+1}$, $x \in X$ and $t \in \{1, ..., T\}$. Together, equations (8) and (9) bring about the following axiom for stationarity of risk attitude in a setting with a finite planning horizon.

A8 (risk stationarity) For all $t \in \{1, ..., T-1\}$ and $x \in X$:

$$\tfrac{1}{2}(\mathsf{x}, x) + \tfrac{1}{2}(\mathsf{x}', x) \succeq_{t} (\mathsf{x}'', x) \;\;\Leftrightarrow\;\; \tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}' \succeq_{t+1} \mathsf{x}'' \quad \forall\, \mathsf{x}, \mathsf{x}', \mathsf{x}'' \in \mathsf{X}^{t+1}.$$

The decision maker ranks lotteries of the form $\frac{1}{2}\mathsf{x} + \frac{1}{2}\mathsf{x}'$ the same way when they are faced in period t as when they are faced in period $t+1$. When facing them in period t, the additional outcome x at the end of the planning horizon, which coincides for all consumption paths, does not change his ranking.

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[17] Observe that, in difference to equation (2), equation (8) has to hold in all periods $t \in \{1, ..., T\}$. However, by time consistency A4 all the requirements in equation (8) can be carried over into the first two periods (see subsection 5.2). The important characteristic of the requirement is that uncertainty resolves in all the different periods.

5.2 Relation to the Infinite Planning Horizon Setting

I briefly point out the analogous reasoning to yield risk stationarity from the assumption expressed in equation (8) in the case of an infinite planning horizon. Denote the consumption paths corresponding to (x, x) and (x', x) by $\mathrm{x}^\infty, \mathrm{x}'^\infty \in X^\infty$. In the infinite horizon setting it is $\succeq_{1|T+1}=\succeq_{1|\infty}=\succeq_{1|T}$. Then, by time consistency, equation (8) for $t = 1$ is equivalent to

$$\tfrac{1}{2}\mathrm{x}^\infty + \tfrac{1}{2}\mathrm{x}'^\infty \succeq_{1|\infty} \mathrm{x}''^\infty \quad \Leftrightarrow \quad (x_1, \tfrac{1}{2}\mathrm{x}^\infty + \tfrac{1}{2}\mathrm{x}'^\infty) \succeq_{1|\infty} (x_1, \mathrm{x}''^\infty)$$

for all $\mathrm{x}^\infty, \mathrm{x}'^\infty, \mathrm{x}''^\infty \in X^\infty$ and $x_1 \in X$. Similarly for $t = 2$ equation (8) is equivalent to

$$(x_1, \tfrac{1}{2}\mathrm{x}^\infty + \tfrac{1}{2}\mathrm{x}'^\infty) \succeq_{1|\infty} (x_1, \mathrm{x}''^\infty) \quad \Leftrightarrow \quad (x_1, x_2, \tfrac{1}{2}\mathrm{x}^\infty + \tfrac{1}{2}\mathrm{x}'^\infty) \succeq_{1|\infty} (x_1, x_2, \mathrm{x}''^\infty)$$

for all $\mathrm{x}^\infty, \mathrm{x}'^\infty, \mathrm{x}''^\infty \in X^\infty$ and $x_1, x_2 \in X$. The latter statement for $t = 2$ can be transformed using the corresponding statement for $t = 1$ into the requirement:

$$\tfrac{1}{2}\mathrm{x}^\infty + \tfrac{1}{2}\mathrm{x}'^\infty \succeq_{1|\infty} \mathrm{x}''^\infty \quad \Leftrightarrow \quad (x_1, x_2, \tfrac{1}{2}\mathrm{x}^\infty + \tfrac{1}{2}\mathrm{x}'^\infty) \succeq_{1|\infty} (x_1, x_2, \mathrm{x}''^\infty)$$

for all $\mathrm{x}^\infty, \mathrm{x}'^\infty, \mathrm{x}''^\infty \in X^\infty$ and $x_1, x_2 \in X$. By induction one obtains the general requirement

$$\tfrac{1}{2}\mathrm{x}^\infty + \tfrac{1}{2}\mathrm{x}'^\infty \succeq_{1|\infty} \mathrm{x}''^\infty \quad \Leftrightarrow \quad (\mathrm{x}^t, \tfrac{1}{2}\mathrm{x}^\infty + \tfrac{1}{2}\mathrm{x}'^\infty) \succeq_{1|\infty} (\mathrm{x}^t, \mathrm{x}''^\infty) \tag{10}$$

for all $\mathrm{x}^\infty, \mathrm{x}'^\infty, \mathrm{x}''^\infty \in X^\infty$, $t \in \mathrm{I\!N}$ and $\mathrm{x}^t \in X^t$. A corresponding[18] axiom for stationarity of risk attitude is found in Chew & Epstein (1991, 356).

5.3 The Risk Stationary Representation

Preference stationarity for the evaluation of uncertain scenarios together with assumptions A1-A5 yield the following representation.

Theorem 3: Choose a nondegenerate closed interval $U^* \subset \mathrm{I\!R}_+$ and let a sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfy axiom A0.

The sequence $\succeq$ satisfies axioms A1-A5 and risk stationarity A8, if and only if, there exist a continuous and surjective function $u : X \to U^*$, a discount factor $\beta \in \mathrm{I\!R}_{++}$, and $\xi \in \mathrm{I\!R}$, such that defining the functions $\tilde{u}_t : \tilde{X}_t \to \mathrm{I\!R}$ for $t \in \{1, ..., T\}$ by

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[18] In difference to the above formulation, the authors require condition (10) for all lotteries, not just for the probability a half ('coin toss') combinations that I have used and which prove sufficient in my setting.

$\tilde{u}_T(x_T) = u(x_T)$ and recursively

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = u(x_{t-1}) + \beta\, \mathcal{M}^{\exp^\xi}(p_t, \tilde{u}_t)$$

the representing equivalence

$$p_t \succeq_t p'_t \quad \Leftrightarrow \quad \mathcal{M}^{\exp^\xi}(p'_t, \tilde{u}_t) \geq \mathcal{M}^{\exp^\xi}(p'_t, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t$$

holds for all periods $t \in \{1, ..., T\}$.
Moreover, given U^* the function u is determined uniquely as are the measures of intertemporal risk aversion $\text{AIRA}^*_t = -\xi$ and $\text{RIRA}^*_t = -\xi\,\text{id}$.

Under risk stationarity the functions f_t characterizing uncertainty aggregation in theorem 1 coincide for different periods. Moreover, the axiom implies that absolute intertemporal risk attitude is constant over outcomes, implying the functional form $f_t(z) = \exp(\xi z)$. The parameter $-\xi$ measures the decision maker's degree of absolute intertemporal risk aversion. These two implications of axiom A8 can be associated with equations (8) and (9) respectively. While it is the idea underlying equation (8) that causes uncertainty aggregation to coincide in different period, it is the assumption of 'coinciding last outcome independence' (9) that is mostly responsible for constant absolute intertemporal risk aversion. Recall from page 15 in section 4.2 that AIRA^*_t measures intertemporal risk aversion in terms of *current* welfare. This measurement with respect to the undiscounted welfare scale is crucial for making intertemporal risk aversion constant over time and to understand the relation to the representation derived in the next section.

6 Timing Indifference

Recursive utility models, as employed in the preceding sections, generally imply an intrinsic preference for early or late resolution of uncertainty. Motivated by a normative perspective on rational decision making, I analyze the consequences of imposing intrinsic timing indifference. First I state and discuss the assumption, then I give the representation.

6.1 Indifference to the Timing of Risk Resolution

A rational decision maker generally prefers an early resolution of uncertainty whenever it permits refining later choices and improve outcomes (or outcome probabilities). Such a preference for an early resolution is instrumental in giving rise to better outcomes (in expectation). Kreps & Porteus (1978) have shown that, in addition, general recursive utility models also allow for an intrinsic preference for an early or late resolution of uncertainty. Here, a decision maker prefers early (or late) resolution of uncertainty, even if the obtained information does not allow to change future outcomes (or outcome probabilities). In Traeger (2007*a*) I analyze the relation between intertemporal risk aversion and an intrinsic preference for the timing of uncertainty resolution. From a normative perspective of a public decision maker, or from a perspective of rational decision making, it can be argued for indifference to the timing of uncertainty resolution. Such a reasoning is motivated by the fact that a decision maker who is non-indifferent is willing to give up welfare for advancing or postponing information of which he knows that it has no effect on his future behavior or payoffs.[19] The timing indifference axiom formally writes as follows.

A9 (indifference to the timing of risk resolution)
For all $t \in \{1, ..., T-1\}$, $x_t \in X$, $p_{t+1}, p'_{t+1} \in P_{t+1}$ and $\lambda \in [0,1]$ it holds

$$\lambda(x_t, p_{t+1}) + (1-\lambda)(x_t, p'_{t+1}) \quad \sim_t \quad (x_t, \lambda p_{t+1} + (1-\lambda)p'_{t+1}).$$

A decision maker, who is indifferent to the timing of risk resolution, does not distinguish between a lottery where the uncertainty about the future faced in period $t+1$ (i.e. whether he faces the future p_{t+1} or p'_{t+1}) resolves in period t (lottery on the left) or in period $t+1$ (lottery on the right). This uncertainty about the future faced in period $t+1$ is described by the probability mixture λ and $1-\lambda$. Note that axiom 9 is always satisfied in the intertemporally additive expected utility standard model.

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[19] For a more detailed discussion on this point I refer to Traeger (2007*a*). Note that a frequent justification of non-trivial timing preference is based on the argument that only recursive models featuring non-indifference to the timing of risk resolution allow to disentangle Arrow Pratt risk aversion from the elasticity of intertemporal substitutability. However, while this reasoning is true for the widespread generalized isoelastic model, it is not true in general (Traeger 2007*b*). Any model of non-trivial intertemporal risk aversion, including the upcoming representation, allows such a disentanglement of Arrow Pratt risk aversion and intertemporal substitutability. The same argument applies for modeling correlation aversion.

6.2 The Timing Indifferent Representation

The assumption of indifference to the timing of risk resolution implies that the evaluation of uncertainty can be reduced to the evaluation of atemporal lotteries expressing uncertainty as probability measures over consumption paths. I denote these standard lotteries by $p_t^{\mathsf{X}}, p_t'^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$. The same reason permits to restrict the definition of uncertainty aggregation rules to the subdomain $\mathsf{X}^t \subset \tilde{X}_t$ yielding functionals $\mathcal{M}^{f_t} : \Delta(\mathsf{X}^t) \times \mathcal{C}^0(\mathsf{X}^t) \to \mathrm{I\!R}$. Uncertainty aggregation rules now take as inputs lotteries p_t^{X} and evaluations $\tilde{u}_t : \mathsf{X}^t \to \mathrm{I\!R}$ of consumption *paths*. In Traeger (2007*a*) I show how the probability measures $p_t^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$ can be derived from their recursive counterparts $p_t \in P_t$ by ‘integrating out’ the information on the timing of risk resolution. This relation, however, is only needed to axiomatize the representation within the more general setting. For an application of the representation theorem, it is sufficient to describe the uncertain future directly by the measures $p_t^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$.

Theorem 4: Choose a nondegenerate closed interval $U^* \subset \mathrm{I\!R}_+$ and let a sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfy axiom A0.

The sequence $\succeq$ satisfies axioms A1-A5, certainty stationarity A6 and timing indifference A9, if and only if, there exists a continuous and surjective function $u : X \to U^*$, a discount factor $\beta \in \mathrm{I\!R}_{++}$ and $\xi \in \mathrm{I\!R}$, such that defining the functions $\tilde{u}_t : \tilde{X}_t \to \mathrm{I\!R}$ for $t \in \{1, ..., T\}$ by

$$\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} u(\mathsf{x}^t_\tau)$$

the representational equivalence

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{\exp^\xi}(p_t^{\mathsf{X}}, \tilde{u}_t) \geq \mathcal{M}^{\exp^\xi}(p_t'^{\mathsf{X}}, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t \tag{11}$$

holds for all periods $t \in \{1, ..., T\}$.

Moreover, given U^* the function u is determined uniquely as are the measures of intertemporal risk aversion $\mathrm{AIRA}^*_t = -\beta^{t-1}\,\xi$ and $\mathrm{RIRA}^*_t = -\beta^{t-1}\,\xi\,\mathrm{id}$.

The representation evaluates outcomes in all periods with the stationary utility function u. A consumption path is evaluated by the discounted sum of per period utility. To evaluate an uncertain future, the decision maker weights the aggregate welfare of the possible consumption paths with their respective probabilities, and applies the uncertainty aggregation rule $\mathcal{M}^{\exp^{\xi}}$. As in the risk stationary setting, uncertainty evaluation takes on the constant absolute intertemporal risk aversion form. For a decision maker starting out directly with an atemporal (standard) lottery description, equation (11) in particular implies

$$p_t^{\mathsf{X}} \succeq_t {p'}_t^{\mathsf{X}} \;\Leftrightarrow\; \mathcal{M}^{\exp^{\xi}}(p_t^{\mathsf{X}}, \tilde{u}_t) \geq \mathcal{M}^{\exp^{\xi}}({p'}_t^{\mathsf{X}}, \tilde{u}_t) \quad \forall\, p_t^{\mathsf{X}}, {p'}_t^{\mathsf{X}} \in \Delta(\mathsf{X}^t)\,.$$

The model coincides with the widespread discounted utility framework for the evaluation of individual consumption paths, but generally employs nonlinear uncertainty aggregation over the welfare generated by the different paths. Note that under the above assumption of axiom A9, a result by Chew & Epstein (1989, 110) permits replacing the independence axiom A3 by a collection of weaker assumptions.

While absolute intertemporal risk aversion in theorem 4 is again constant over outcomes, it changes over time. For an intuition why the coefficient of intertemporal risk aversion is discounted it is helpful to consider a decision maker facing a lottery over outcomes in period T. First, let the lottery and the risk evaluation take place in period t with $1 < t < T$.[20] The decision maker assigns a utility to each of the possible lottery outcomes[21] and aggregates over uncertainty according to his intertemporal risk attitude in period t. Second, let the lottery and the decision maker's risk evaluation take place in period $t-1$. Then, the utilities attached to the lottery outcomes are discounted by the factor β. In consequence, also the welfare *variation* corresponding to the risky utility change *is discounted.* If the decision maker's coefficient of *absolute intertemporal risk aversion* were the same in period $t-1$ as in period t, the smaller welfare variation going along with the lottery would imply a smaller effective risk aversion.[22] Then, the earlier he faced and evaluated the lottery, the less his effective risk aversion and the higher would be the certainty equivalent. The decision maker would exhibit an intrinsic preference for early resolution of risk. In consequence, for a timing indifferent decision maker, the coefficient

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[20] The lottery, e.g. flipping of a coin, takes place in period t and determines the outomes, e.g. consumption levels, in period T.

[21] More precisely, he assigns utilities to each of the consumption paths from t until T differing only in their period T outcomes.

[22] The reasoning crucially depends on the fact that the decision maker evaluates uncertainty with constant absolute intertemporal risk aversion. For an intuition, why axiom A9 implies constant absolute intertemporal risk aversion see Traeger (2007*a*).

of absolute intertemporal risk aversion in period $t-1$ must be, in absolute terms, bigger than in period t and AIRA^*_t must be discounted with respect to AIRA^*_{t-1}.

Recall from page 15 in section 4.2 that AIRA^*_t measures intertemporal risk aversion in terms of a 'current value' measure scale for welfare. Adopting instead a 'present value' measure scale of welfare, as in Traeger (2007*a*) makes the coefficient of absolute intertemporal risk aversion constant over time. From such a present value perspective, the following reasoning describes the intertemporal risk attitude of a timing indifferent decision maker who discounts future welfare. Postponing a lottery further into the future implies decreasing absolute welfare *variation* in present value terms. The smaller welfare variation causes a smaller welfare loss produced by the uncertainty. The tension between such a risk attitude and risk stationarity is discussed in the next section.

7 Implications for Discounting

The section elaborates the consequences of the derived results for long-term evaluation. First, combining the assumptions of risk stationarity and indifference to the timing of risk resolution implies a zero rate of pure time preference. Second, an intertemporal risk averse decision maker gives less weight to the future the less he knows about the consequences of his (in-)actions.

7.1 The Pure Rate of Time Preference

A decision maker who satisfies risk stationarity ranks lotteries the same way, independent of the amount of periods they lie in the future. Therefore, his intertemporal risk aversion measured with respect to 'current value' welfare gains and losses has to be constant. A decision maker who satisfies indifference with respect to the timing of risk resolution ranks lotteries over outcomes in a fixed period the same way, independent of the period in which the risk resolves. To these ends, the decision maker's intertemporal risk aversion measured in 'present value' terms has to be constant over time. The current value measures AIRA^*_t pick up the discount rate, reflecting that the decision maker cares less for a given welfare variation, the further it lies in the future.

In consequence, there is only one situation where a devaluation of the future is compatible with both axioms. For a decision maker who is intertemporal risk neutral neither the assumption of risk stationarity nor that of timing indifference have bite.[23] The coefficient $\text{AIRA}_t = 0$ for all $t \in \{1, ..., T\}$ and, therefore, is independent of β. For a nontrivial model of intertemporal risk averse decision making the following result obtains.

Theorem 5: A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfying axiom A0, satisfies A1-A5, strict intertemporal risk aversion A7$^{\text{s}}$, risk stationarity A8, and timing indifference A9, if and only if, there exists a representation in the sense of theorem 4 with $\xi < 0$ and $\beta = 1$.

A decision maker who accepts the above axioms does not discount the future for reasons of pure time preference.

7.2 Discounting for Reasons of Uncertainty

Instead of (or in addition to) a pure rate of time preference, however, an intertemporal risk averse decision maker effectively gives less weight to future welfare if uncertainty increases over time. For the simplest presentation of this reasoning I assume a preference representation in the sense of theorem 5 where $\beta = 1$ and fix $U^* = [0, 1]$. Moreover, let $p_1^{\mathsf{X}} \in \Delta(\mathsf{X}^1)$ be a product measure

$$p_1^{\mathsf{X}} = \mathrm{p}_1 \otimes ... \otimes \mathrm{p}_T \text{ with } \mathrm{p}_t \in \Delta(X) \tag{12}$$

for $t \in \{1, ..., T\}$, implying that outcomes in different periods are independently distributed. Then, the evaluation of lottery p_1^{X} simplifies to the form

$$\mathcal{M}^{\exp^\xi}\left(p_1^{\mathsf{X}}, \tilde{u}_t\right) = \mathcal{M}^{\exp^\xi}\left(\mathrm{p}_1, u\right) + \mathcal{M}^{\exp^\xi}\left(\mathrm{p}_2, u\right) + ... + \mathcal{M}^{\exp^\xi}\left(\mathrm{p}_T, u\right). \tag{13}$$

Let p_t^u denote the pushforward of p_t under u, i.e. the welfare lottery induced by the uncertainty p_t over outcomes x_t in period t. In order to define increasing uncertainty over time, I employ Rothschild & Stiglitz's (1970) definition of increasing risk. Random welfare u_{t+1} in period $t + 1$ is said to be more uncertain than the random welfare u_t in period t, if the corresponding probability distribution p_{t+1}^u has more weight on the tails

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[23] Precisely, risk stationarity has no additional bite with respect to the assumption of certainty stationarity.

than p_t^u.[24] Formally, let P_t denote the cumulative distribution function characterizing the measure p_t^u for $t \in \{1, ..., T\}$. Then, increasing uncertainty of welfare over time is defined by the requirement

$$\textstyle\int_0^z \mathrm{P}_{t+1}(z') - \mathrm{P}_t(z')\, dz' \geq 0 \text{ for all } z \in [0,1] \tag{14}$$

$$\textstyle\int_0^{z^*} \mathrm{P}_{t+1}(z') - \mathrm{P}_t(z')\, dz' > 0 \text{ for some } z^* \in [0,1], \text{ and} \tag{15}$$

$$\textstyle\int_0^1 \mathrm{P}_{t+1}(z') - \mathrm{P}_t(z')\, dz' = 0 \tag{16}$$

for all $t \in \{1, ..., T-1\}$. Equation (16) implies that expected welfare is constant over time, i.e. it exits $\bar{u} \in [0,1]$ such that $\mathrm{E}_{\mathrm{p}_t} u(\mathrm{x}_t) = \bar{u}$ for all $t \in \{1, ..., T\}$. The condition is, in particular, satisfied for a mean preserving spread. The following proposition holds.

Proposition 2: Let preferences be described by representation theorem 5. Let future outcomes be described by $p_1^{\mathbf{X}}$ as defined in equation (12) exhibiting increasing uncertainty over time in the sense of equations (14)-(16). Then

$$\mathcal{M}^{\exp^\xi}\big(\mathrm{p}_{t+1}, u\big) < \mathcal{M}^{\exp^\xi}\big(\mathrm{p}_t, u\big) \text{ for all } t \in \{1, ..., T-1\}\ . \tag{17}$$

Even if expected welfare is the same in all periods and the decision maker has no pure time preference the certainty equivalent welfare in future periods, i.e. the summands in equation (13), constitutes a decreasing sequence over time. Under increasing uncertainty, intertemporal risk aversion can discount future welfare similar to a utility discount factor (pure time preference). In particular, even for a decision maker employing an infinite planning horizon, the evaluation functional does not necessarily diverge when adopting a zero rate of pure time preference.

7.3 Discussion on Discounting

I point out three different perspectives on the results found in this section and briefly relate them to empirical findings by Epstein & Zin (1991). First, from a normative perspective on rational decision making, one can use theorem 5 to argue that the pure

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[24] An equivalent characterization is that the riskier random variable can be obtained from the less risky random variable by adding some noise (Rothschild & Stiglitz 1970). Note that Rothschild & Stiglitz (1970) give a weak definition of increasing risk, while the definition in this paper adds the strict inequality in equation (15).

rate of time preference should be set to zero. Second, from a theoretical point of view, the results point out an interesting tension between the involved axioms, in particular between intertemporal risk aversion, risk stationarity, and timing indifference.[25] Third, from a behavioral perspective, the theorems give straight forward testable models for decision making under uncertainty. In particular, if β is different from 1, a change of absolute intertemporal risk aversion over time immediately answers the question whether decision makers rather violate risk stationarity or timing indifference.

From a descriptive perspective, the lack of pure time preference seems suspicious at first sight. However, note that in an application to asset pricing Epstein & Zin (1991) estimate a discount factor surprisingly close to unity when they allow for disentangled parameters of Arrow Pratt risk aversion and intertemporal substitutability. Epstein & Zin (1991) use the generalized isoelastic model which implicitly assumes constant *relative* intertemporal risk aversion rather then constant absolute intertemporal risk aversion. However, their estimated functional form for intertemporal aggregation, describing the welfare function u, is almost logarithmic. The latter form happens to describe the only overlap between the generalized isoelastic model and the model characterized axiomatically by theorem 3. Thus, a comparison with the results of Epstein & Zin (1991) suggests that neither of the representations characterized in this paper seems to be far off from describing behavior observed in financial markets. This observation might be less surprising when taking into account the reasoning carried out in section 7.2 that intertemporal risk aversion can devaluate expected future welfare. The intertemporal additive expected utility standard model does not allow for intertemporal risk aversion. Therefore, the only possibility it permits to capture a difference in the weighting of expected welfare between different periods is by introducing a positive rate of pure time preference.

The section has discussed two different rationales for discounting, pure time preference and intertemporal risk aversion in combination with increasing uncertainty. While both imply a reduced weight given to future expected welfare, there are two important differences. First, in a stationary evaluation, a positive pure rate of time preference implies exponential discounting. Under intertemporal risk aversion, the form of discounting will generally depend on risk aversion and the way uncertainty increases over time. Second,

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[25] Recall that Chew & Epstein (1989, 110) have shown that in the context of axiom A9 the independence axiom can be replaced by a collection of significantly weaker axioms, making it a less likely to be violated (on its own).

and possibly most important, the two discounting rationales only work into the same direction as long as the increase in uncertainty is exogenous. As soon as uncertainty becomes endogenous to decision making, both discounting rationales can yield very different lottery comparisons. Then, a choice that reduces uncertainty puts more weight on the future and gains more overall weight. Thus, it will be preferred. As I point out in the subsequent subsection for the example of global climate change, this effect can trigger an opposite effect for policy recommendations than does a positive pure rate of time preference.

7.4 A Conjecture Concerning Climate Change Evaluation

I close by pointing out an important application of the modeling framework derived in this section. In relation to climate change, the 'correct' discount rate for long-term evaluation is hotly and prominently debated following the report on climate change carried out by Sir Nicholas Stern (2007) on behalf of the British Government. The Stern review employs a close to zero rate of pure time preference based mostly on ethical arguments. This choice has been criticized and defended in numerous recent articles, among them Nordhaus (2007), Weitzman (2007), and Arrow (2007). This debate on the appropriate discount rate in climate change evaluation reaches back to Nordhaus' (1993,1994) integrated assessment model for climate change and its critical discussions (see e.g. Toth 1995, Plambeck, Hope & Anderson 1997). In particular, Plambeck et al. (1997, 85) point out that a reduction of the pure rate of time preference from 3%, as assumed by Nordhaus (1993), to 0% (corresponding to $\beta = 1$), would result in an optimal abatement path that cuts emissions by 50% from the baseline to the year 2100, as opposed to 10% in the assessment of Nordhaus (1993).

Theorem 5 states formal axioms dealing with consistency aspects of evaluation under uncertainty that imply a zero rate of pure time preference. Proposition 2 points out that (at least some part of) observed discounting usually attributed to pure time preference can be also be explained by a combination of intertemporal risk aversion and increasing uncertainty over time. An evaluation of climate change in the framework derived in theorem 5 replaces pure time preference by intertemporal risk aversion. Here, uncertainty has a higher cost than in the standard models used for climate change evaluation, which apply the intertemporal risk neutral standard framework (if they consider uncertainty at all).[26] In consequence, an evaluation of global climate change under the assumptions of theorem 5 implies an additional preference for scenarios that give rise to a less uncertain future. Since uncertainty is likely to increase in the perturbation of the climate system, which increases with the amount of greenhouse gas emissions, a first conjecture is that the additional effect caused by intertemporal risk aversion yields an even higher abatement recommendation than with just a zero rate of pure time preference. A closer analysis of the conjecture constitutes an interesting area of future research in climate change evaluation.

8 Conclusions

The paper has introduced axioms characterizing a stationary scenario evaluation for decision makers with a finite planning horizon. The axioms offer an alternative to the standard stationarity axioms that rely on an infinite time horizon and a positive rate of pure time preference. I have derived the general stationary evaluation frameworks that allow for non-trivial intertemporal risk attitiude. Certainty stationarity implies the standard discounted utility model on certain consumption paths. For risk stationary preferences, the decision maker moreover exhibits constant absolute intertemporal risk aversion over outcomes. When measured with respect to 'current value' welfare gains and losses, the degree of absolute intertemporal risk aversion is constant over time.

The recursive evaluation employs Kreps & Porteus' (1978) framework of temporal lotteries. Here, the description of an uncertain scenario comprises information on the timing of risk resolution. An assumption of indifference to the timing of risk resolution in situations where the obtained information cannot be used to alter outcomes brings the modeling framework back into the standard setting, where the uncertainty relevant for decision making is described by probability measures over consumption paths. Also the assumption of timing indifference makes absolute intertemporal risk aversion constant over outcomes. However, it requires absolute intertemporal risk aversion to fall over time at the rate of pure time preference.

Together, the axioms of risk stationarity and indifference to the timing of risk resolution imply a zero rate of pure time preference for decision makers with a non-trivial

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[26] An exception is a stylized simulation by Ha-Duong & Treich (2004) that features two possible damage states in a generalized isoelastic framework.

intertemporal risk attitude. In particular, the result identifies the much contended impatience parameter as a degeneracy of intertemporal risk neutral decision making. For the intertemporal risk averse decision maker I have pointed out a different rationale for discounting. If uncertainty is increasing over time, he values expected future utility less than current utility, also with a zero rate of pure time preference. If uncertainty increases exogenously, the effect resembles standard discounting (with a not necessarily constant discount rate). However, if uncertainty is endogenous to decision making, the different rationale for discounting can imply opposite effects on evaluation. As a prominent example, I have pointed out how in the context of climate change the more comprehensive evaluation framework is likely to yield a significantly higher mitigation recommendation.

The modeling framework developed here is applicable in any field of economics where time and uncertainty play an important role. Two applications are of particular interest. The first is a comparison with the generalized isoelastic model applied to asset pricing. The disentanglement of intertemporal substitutability and Arrow Pratt risk aversion in the generalized isoelastic model allows a partial explanation of the equity premium puzzle (Epstein & Zin 1991, Mehra & Prescott 2003). In the light of intertemporal risk aversion, this observation is explained by relaxing the assumption of intertemporal risk neutrality and allowing for risk aversion with respect to utility gains and losses. However, the current paper suggests that intertemporal risk aversion should be constant in absolute terms rather than in relative terms as it is implicitly assumed in the generalized isoelastic model. In the same context, it would be interesting to determine whether the near zero rate of pure time preference estimated by Epstein & Zin (1991) for the generalized isoelastic model prevails. A second and related application concerns the welfare cost of volatility. Since Lucas (1987) it is known that the standard model favors policy measures fostering additional growth at the expense of higher volatility. This conclusion is likely to change in the more general setting with intertemporal risk aversion.

Appendix

A Proofs for Section 3

Proof of theorem 1: The proof builds on representation theorem 2 in Traeger (2007*b*) for non-stationary preferences. I show that axiom A6 allows picking a coinciding utility function evaluating outcomes in different periods and translate the recursive description of aggregate utility into the ('current value') form given in the theorem.

Sufficiency: For a given $a \in \mathbb{R}_{++}$, the proof makes use of the notation $\mathsf{A}^a = \{\mathsf{a}^a \in \mathsf{A} : \mathsf{a}^a(z) = a\,z + b\,,\, b \in \mathbb{R}\}$. Moreover, I define for a given sequence of continuous functions $u_t : X \rightarrow \mathbb{R}$ with range U_t, $t \in \{1, ..., T\}$, the normalization constants ϑ_t by setting $\vartheta_T = 0$ and for $t < T$

$$\vartheta_t = \frac{\overline{U}_{t+1}\underline{U}_t - \underline{U}_{t+1}\overline{U}_t}{\overline{U}_t - \underline{U}_t}\,.$$

By axioms A1-A5, corollary 3 in Traeger (2007*b*) implies the existence of the following preference representation for $\succeq$. There exist continuous functions $u_t : X \rightarrow \mathbb{R}$ with range U_t and strictly increasing and continuous functions $f_t^* : \mathbb{R} \rightarrow \mathbb{R}$ such that defining recursively the aggregate welfare functions $\tilde{u}_t^* : \tilde{X}_t \rightarrow \mathbb{R}$ by $\tilde{u}_T^*(x_T) = u(x_T)$ and

$$\tilde{u}_{t-1}^*(x_{t-1}, p_t) = u_{t-1}(x_{t-1}) + \mathcal{M}^{f_t^*}(p_t, \tilde{u}_t^*) + \frac{\vartheta_{t-1}}{\theta_t}$$

the expression $\mathcal{M}^{f_t^*}(p_t, \tilde{u}_t^*)$ represents the lotteries p_t in the sense of equation (6). Moreover, the sequences of tuples $(u_t, f_t^*)_{t\in\{1,...,T\}}$ and $(u_t', f_t^{*\prime})_{t\in\{1,...,T\}}$ both represent $\succeq$, if and only if, for some $a \in \mathbb{R}_{++}$ there exist affine transformations $\mathsf{a}_t \in \mathsf{A}$ and $\mathsf{a}_t^a \in \mathsf{A}^a$ for all $t \in \{1, ..., T\}$ such that $(u_t', {f^*}_t|_{\tilde{U}_t}) = (\mathsf{a}_t^a\; u_t\,,\; \mathsf{a}_t f_t^{*\prime}|_{\tilde{U}_t'}\; \mathsf{a}_t^{a-1}\;)$.

To translate axiom A6 into the representation stated above recall that $\mathcal{M}^{f_\tau^*}(\tilde{x}_\tau, \tilde{u}_\tau^*) = \tilde{u}_\tau^*(\tilde{x}_\tau)$, recursively yielding for $\mathsf{x} \in \mathsf{X}^t$ that $\mathcal{M}^{f_t^*}(\mathsf{x}, \tilde{u}_t^*) = \sum_{\tau=t}^T u_\tau(\mathsf{x}_\tau) + c_t$ for some constants $c_t \in \mathbb{R}$. Therefore, axiom A6 with $\mathsf{x} = (\mathsf{x}_2, ..., \mathsf{x}_T)$ translates into the requirement

$$\begin{aligned} \textstyle\sum_{\tau=2}^T u_{\tau-1}(\mathsf{x}_\tau) + \cancel{u_T(x)} &\geq \textstyle\sum_{\tau=2}^T u_{\tau-1}(\mathsf{x}'_\tau) + \cancel{u_T(x)} \\ \Leftrightarrow \textstyle\sum_{\tau=2}^T u_\tau(\mathsf{x}_\tau) \qquad\qquad &\geq \textstyle\sum_{\tau=2}^T u_\tau(\mathsf{x}'_\tau)\,. \end{aligned}$$

for all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^2$. The above equivalence implies that $\sum_{\tau=2}^T u_\tau(\mathsf{x}_\tau)$ and $\sum_{\tau=2}^T u_{\tau-1}(\mathsf{x}_\tau)$ are both representations for $\succeq_2 |_{\mathsf{X}^2}$. In consequence, by the uniqueness result stated above, there exist $a \in \mathbb{R}_{++}$ and $b_t \in \mathbb{R}, t \in \{1, ..., T-1\}$, such that $u_t = a\,u_{t+1} + b_t$ for all

$t \in \{1, ..., T-1\}$.[27] The freedom in the uniqueness of $(u_t)_{t\in\{1,\ldots,T\}}$ can be used to eliminate the affine displacement parameter b_t by defining $u'_t = u_t - \sum_{\tau=t}^{T-1} a^{\tau-t} b_\tau$ for $t \in \{1, ..., T-1\}$. For u'_t find that $u'_t = u_t - \sum_{\tau=t}^{T-1} a^{\tau-t} b_\tau = a u_{t+1} + b_t - b_t - a \sum_{\tau=t+1}^{T-1} a^{\tau-t} b_\tau = a u'_{t+1}$. Letting $u = u'_1$ and $\beta = a^{-1}$ yields $u_t = \beta^{t-1} u$. The change of the representing utility functions from u_t to u'_t corresponds to the affine transformations $u'_t = \mathfrak{a}^1_t \, u_t$ with $\mathfrak{a}^1_t(z) = z - \sum_{\tau=t}^{T-1} a^{\tau-t} b_\tau$ and requires an according transformation of the representing functions f^*_t to $f^{*\prime}_t = f^*_t \, \mathfrak{a}^{1\;-1}$. Then, the sequence of tuples $(\beta^{t-1}u, f^{*\prime}_t)_{t\in\{1,\ldots,T\}}$ represents $\succeq$.

The normalization constants ϑ_t for the representing tuples $(\beta^{t-1}u, f^{*\prime}_t)_{t\in\{1,\ldots,T\}}$ are

$$\vartheta_t = \frac{\overline{U}_{t+1}\underline{U}_t - \underline{U}_{t+1}\overline{U}_t}{\overline{U}_t - \underline{U}_t} = \frac{\beta^t\overline{U}\,\beta^{t-1}\underline{U} - \beta^t\underline{U}\,\beta^{t-1}\overline{U}}{\beta^{t-1}(\overline{U}-\underline{U})} = 0\,.$$

Using the tuples $(\beta^{t-1}u, f^{*\prime}_t)_{t\in\{1,\ldots,T\}}$ in the representation therefore yields the following recursive definition of aggregate utility

$$\tilde{u}^*_t(\cdot,\cdot) = u'_t(\cdot) + \mathcal{M}^{f^{*\prime}_{t+1}}(\cdot, \tilde{u}^*_{t+1}) + 0 = \beta^{t-1}u(\cdot) + \mathcal{M}^{f^{*\prime}_{t+1}}(\cdot, \tilde{u}^*_{t+1})\,.$$

which is equivalent to

$$\beta^{1-t}\,\tilde{u}^*_t(\cdot, p_{t+1}) = u(\cdot) + \beta\;\beta^{1-(t+1)} f^{*\,\prime\;-1}_{t+1}\left(\mathrm{E}_{p_{t+1}} f^{*\,\prime}_{t+1}\left(\beta^{(t+1)-1}\beta^{1-(t+1)}\;\tilde{u}^*_{t+1}\right)\right).$$

Defining $\tilde{u}_t = \beta^{1-t}\tilde{u}^*_t$ and $f_t(z) = f^{*\prime}_t(\beta^{t-1}z)$ for all t yields the representation

$$\tilde{u}_t(\cdot, p_{t+1}) = u(\cdot) + \beta\, f^{-1}_{t+1}\Big(\mathrm{E}_{p_{t+1}} f_{t+1} \circ \tilde{u}_{t+1}\Big)$$

stated in the theorem.

Necessity: Axioms A1-A5 follow immediately from the necessity part in theorem 2 in Traeger (2007*b*). To see that axiom A6 is always satisfied in the representation observe that $\mathcal{M}^{f_\tau}(\tilde{x}_\tau, \tilde{u}_\tau) = \tilde{u}_\tau(\tilde{x}_\tau)$ and recursively $\mathcal{M}^{f_t}(\mathsf{x}, \tilde{u}_t) = \sum_{\tau=t}^{T} \beta^{\tau-t}\, u(\mathsf{x}_\tau)$. Then, axiom A6 is seen to hold by verifying the following equivalence

$$\begin{aligned} &\textstyle\sum_{\tau=1}^{T-1} \beta^{\tau-1}\, u(\mathsf{x}_{\tau+1}) + \cancel{\beta^{T-1} u(x)} \geq \sum_{\tau=1}^{T-1} \beta^{\tau-1}\, u(\mathsf{x}'_{\tau+1}) + \cancel{\beta^{T-1} u(x)} \\ \Leftrightarrow &\textstyle\sum_{\tau=2}^{T} \beta^{\tau-2}\, u(\mathsf{x}_\tau) \qquad\qquad\qquad\quad \geq \sum_{\tau=2}^{T} \beta^{\tau-2}\, u(\mathsf{x}'_\tau)\,. \end{aligned}$$

Uniqueness: The representation is a special case of the representation in the sense of corollary 3 in Traeger (2007*b*), stated at the beginning of this proof. Here, $u_t = \beta^{t-1}\, u$ and $f^*_t = f_t \circ \mathfrak{a}$ for some $\mathfrak{a} \in \mathsf{A}$. Thus, the uniqueness result follows immediately from corollary 3 in Traeger (2007*b*). □

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[27] Here it is $u'_t = u_{t+1}$. Coincidence of the representations (only) on the certain outcome paths is enough to assure the uniqueness result for $(u_t)_t \in \{1, ..., T\}$.

B Proofs for Section 4

Proof of theorem 2: The proof translates axiom A7$^{\mathrm{s}}$ into the representation of theorem 1. Using the premise, I translate the second line of the axiom into a concavity condition for f_t. Precisely, the resulting equation (21) would be a straight forward concavity condition, if it had hold for all convex combinations in the argument of f_t. However, the premise only requires equation (21) to hold for a limited set of convex combinations. The proof that the limited requirement still yields concavity of the functions f_t works, with one exception, analogous to the according proof for the non-stationary analysis carried out in Traeger (2007*b*). I point out the step that differs and, hereafter, refer the reader to the proof of theorem 3 in Traeger (2007*b*).

The proof is divided into four parts. The first part derives the (restricted) concavity condition. The second part relates the rest of the proof to the one carried out in Traeger (2007*b*). The third part deals with statements b) and c) of the theorem. The fourth part takes care of the necessity of the axioms for weak/strict intertemporal risk aversion/seeking.

Sufficiency: Part I: This part of the proof translates axiom A7$^{\mathrm{s}}$ into the representation of theorem 1. I start with the first line, i.e the premise:

$$\begin{array}{rcl} \bar{\mathsf{x}} & \sim_t & \mathsf{x} \\ \Rightarrow \displaystyle\sum_{\tau=t}^{T} \beta^{\tau-t}\, u(\bar{x}) & = & \displaystyle\sum_{\tau=t}^{T} \beta^{\tau-t}\, u(\mathsf{x}_\tau)\,. \end{array} \tag{18}$$

The existence of $\tau \in \{t, ..., T\}$ such that $(\bar{\mathsf{x}}_{-\tau}, \mathsf{x}_\tau) \not\sim_t \bar{\mathsf{x}}$ translates into

$$u(\mathsf{x}_\tau) \neq u(\bar{x}) \text{ for some } \tau \in \{t, ..., T\}. \tag{19}$$

The second line of axiom A7$^{\mathrm{s}}$ yields

$$\begin{array}{rcl} \bar{\mathsf{x}} & \succ_T & \sum_{i=t}^{T} \frac{1}{T-t+1} \left(\bar{\mathsf{x}}_{-i}, \mathsf{x}_i\right) \\ \Rightarrow \quad \sum_{\tau=t}^{T} \beta^{\tau-t} u(\bar{x}) & > & f_t^{-1}\Big[\sum_{i=t}^{T} \frac{1}{T-t+1} f_t\Big[\sum_{\tau=t}^{T} \beta^{\tau-t} u\left((\bar{\mathsf{x}}_{-i}, \mathsf{x}_i)_\tau\right)\Big]\Big] \\ \Rightarrow f_t\Big[\sum_{\tau=t}^{T} \beta^{\tau-t} u(\bar{x})\Big] & > & \sum_{i=t}^{T} \frac{1}{T-t+1} f_t\Big[\sum_{\tau=t}^{T} \beta^{\tau-t} u\left((\bar{\mathsf{x}}_{-i}, \mathsf{x}_i)_\tau\right)\Big]. \end{array}$$

Using equation (18) the left hand side can be transformed as follows:

$$\begin{aligned} f_t\Big[\sum_{\tau=t}^T \beta^{\tau-t}u(\bar{x})\Big] &= f_t\left[\tfrac{T-t}{T-t+1}\left[\sum_{\tau=t}^T \beta^{\tau-t}u(\bar{x})\right] + \tfrac{1}{T-t+1}\left[\sum_{\tau=t}^T \beta^{\tau-t}u(\mathsf{x}_\tau)\right]\right] \\ &= f_t\left[\tfrac{1}{T-t+1}\left[\sum_{i=t}^T\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i},\mathsf{x}_i)_\tau\right)\right]\right] \\ &= f_t\left[\sum_{i=t}^T \tfrac{1}{T-t+1}\left[\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i},\mathsf{x}_i)_\tau\right)\right]\right] \;, \end{aligned}$$

yielding the inequality

$$\begin{aligned} & f_t\left[\sum_{i=t}^T \tfrac{1}{T-t+1}\left[\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i},\mathsf{x}_i)_\tau\right)\right]\right] \\ > \;\; & \sum_{i=t}^T \tfrac{1}{T-t+1} f_t\Big[\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i},\mathsf{x}_i)_\tau\right)\Big] \,. \end{aligned} \tag{20}$$

Define the function $\tilde{z} : \mathsf{X}^t \rightarrow \mathrm{I\!R}$ by $\tilde{z}(\mathsf{x}) = \sum_{\tau=t}^T \beta^{\tau-t}u\left(\mathsf{x}_\tau\right)$. Restricting the domain to those consumption paths that satisfy condition (19) the function is onto the interior of the interval $\tilde{U}_t$ which I denote by $\Gamma = \big(\sum_{\tau=t}^T \beta^{\tau-t}\underline{U}\,,\, \sum_{\tau=t}^T \beta^{\tau-t}\overline{U}\big)$. For given consumption paths $\bar{\mathsf{x}}, \mathsf{x} \in \mathsf{X}^t$ I define $z_i = \tilde{z}\big((\bar{\mathsf{x}}_{-i}, \mathsf{x}_i)\big)$. In this notation equation (20) becomes

$$f_t\left(\sum_{i=t}^T \tfrac{1}{T-t+1}\, z_i\right) \quad > \quad \sum_{i=t}^T \tfrac{1}{T-t+1} f_t(z_i). \tag{21}$$

If equation (21) had to hold for all $z_i \in \Gamma$ it would be a straight forward condition for strict convexity of f_t. However axiom A7$^{\mathrm{s}}$ does not immediately imply that the equation has to be met for every sequence $(z_i)_{i\in\{t,\ldots,T\}}$ with $z_i \in \Gamma$. Equation (21) only has to hold for sequences $(z_i)_{i\in\{t,\ldots,T\}}$ which are generated by consumption paths $\bar{\mathsf{x}}, \mathsf{x} \in \mathsf{X}^t$ that satisfy the premise of axiom A7$^{\mathrm{s}}$.

Part II: It is left to show that equation (21) implies strict concavity of f_t also if it only has to hold for sequences $(z_i)_{i\in\{t,\ldots,T\}}$ which are generated by consumption paths $\bar{\mathsf{x}}, \mathsf{x} \in \mathsf{X}^t$ that satisfy the premise of axiom A7$^{\mathrm{s}}$. This part of the proof is mostly analogous to the proof of theorem 3 in Traeger (2007*b*), which first shows that the condition implies local strict concavity of f_t, then weak concavity on the entire set Γ and finally strict concavity on Γ.[28] The only difference in the proof is that the stationarity assumption in this paper permits to generate every point $z^o \in \Gamma$ from a constant consumption path, i.e. for all $z^o \in \Gamma$ exists $\bar{\mathsf{x}} \in \mathsf{X}$ such that $z^o = \tilde{z}(\bar{\mathsf{x}})$. This fact is immediate from observing that

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[28] The step proving local strict concavity correspond to parts two of the respective proof. The fact that the functions f_t are not assumed to be differentiable slightly complicates the step from local to global concavity which is shown in part three of the proof of theorem 3 in Traeger (2007*b*).

$\tilde{z}(\bar{\mathrm{x}}) = u\left(\bar{x}\right)\sum_{\tau=t}^{T}\beta^{\tau-t}$ is continuous as a function of $\bar{x}$ and onto Γ. Therefore, for every $z^o \in \Gamma$ a set of local perturbations x around the constant consumption $\bar{\mathrm{x}}$ generating z^o, which satisfy the premise of axiom A7$^{\mathrm{s}}$, can be used to derive local convexity of f_t. In the non-stationary setting not all $z^o \in \Gamma$ can be generated by a constant consumption path. That circumstance slightly complicates the respective analysis and, there, requires a stronger axiom for the definition of intertemporal risk aversion by enlarging the domain on which the axiom of intertemporal risk aversion implies a non-trivial restriction on preferences.

Part III: Assertion b) is obtained by replacing A7$^{\mathrm{s}}$ by A7$^{\mathrm{w}}$and the strict inequaties by their weak counterparts.[29] A decision maker is intertemporal risk neutral if his preferences satisfy weak risk seeking as well as weak risk aversion. Therefore, assertion b) implies that the function $f_t \circ g_t^{-1}$ has to be concave and convex at the same time and, thus, linear.

Necessity: Part IV: Necessity is implied by theorem 3 in Traeger (2007*b*) for $\mathrm{x} = \bar{\mathrm{x}}$. □

Proof of proposition 1: The proposition is an immediate consequence of proposition 4 in Traeger (2007*b*). For part a) observe that fixing the zero level for $u(x^0)$ in the representation of theorem 1 fixes the zero level of u_t for all periods. □

C Proofs for Section 5

Proof of theorem 3: The proof is divided into four parts. In the first part, I translate axiom A8 into the representation of theorem 1 and derive a functional equation for the functions f_t parameterizing uncertainty aggregation. The known solution is translated back into the representation in part two of the proof. Part three shows necessity of the axioms and part four gives uniqueness and calculates the measures of intertemporal risk aversion.

Sufficiency: Part I: First note that axiom A8 implies axiom A6 by choosing $\mathrm{x} = \mathrm{x}'$. Therefore a representation in terms of theorem 1 exists. In the following, I translate axiom A8 for $t \in \{1, ..., T-1\}$ into the latter representation. Note that, by definition of x as

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[29] In this case the second step in part three becomes redundant.

an element of X^{t+1}, the period τ entry of the consumption path $(\mathsf{x}, x) \in \mathsf{X}^t$ corresponds to $(\mathsf{x}, x)_\tau = \mathsf{x}_{\tau+1}$ for $\tau \in \{t, ..., T-1\}$. The left hand side of the equivalence in axiom A8 translates into

$$\begin{aligned}
&\tfrac{1}{2}(\mathsf{x}, x^0) + \tfrac{1}{2}(\mathsf{x}', x^0) && \succeq_t (\mathsf{x}'', x^0) \\
\Leftrightarrow\ & f_t^{-1}\Big\{\tfrac{1}{2} f_t\Big[\textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} u(\mathsf{x}_{\tau+1}) + \beta^{T-t} u(x)\Big] \\
&\quad + \tfrac{1}{2} f_t\Big[\textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} u(\mathsf{x}'_{\tau+1}) + \beta^{T-t} u(x)\Big]\Big\} \geq \Big[\textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} u(\mathsf{x}''_{\tau+1}) + \beta^{T-t} u(x)\Big] .
\end{aligned}$$

For given consumption $\mathsf{x}, \mathsf{x}', \mathsf{x}'' \in \mathsf{X}$ and $x \in X$ I define $S = \sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)}\, u(\mathsf{x}_\tau)$, $S' = \sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)}\, u(\mathsf{x}'_\tau)$, $S'' = \sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)}\, u(\mathsf{x}''_\tau)$ and $A = \beta^{T-t}\, u(x)$. Varying the consumption paths x, x' and x'' in X^{t+1} goes along with varying S, S' and S'' in the interval $[\frac{1-\beta^{T-t}}{1-\beta}\, \underline{U}, \frac{1-\beta^{T-t}}{1-\beta}\, \overline{U}]$. Similarly, as x is varied in X the value A takes on any number in the interval $[\beta^{T-t}\, \underline{U}, \beta^{T-t}\, \overline{U}]$. In the introduced notation, the above inequality corresponding to the left hand side of the equivalence in axiom A8 writes as

$$f_t^{-1}\Big\{\tfrac{1}{2} f_t\big[S + A\big] + \tfrac{1}{2} f_t\big[S' + A\big]\Big\} - A \geq S''. \tag{22}$$

In the same notation the right hand side of the equivalence in axiom A8 translates into

$$f_{t+1}^{-1}\Big\{\tfrac{1}{2} f_{t+1}\big[S\big] + \tfrac{1}{2} f_{t+1}\big[S'\big]\Big\} \geq S''. \tag{23}$$

For every lottery $p_{t+1} \in P_{t+1}$ there exists a certainty equivalent which is a certain consumption path.[30] In consequence, for any choice of $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^{t+1}$ exists a certainty equivalent $\mathsf{x}'' \in \mathsf{X}^{t+1}$ to the lottery $\frac{1}{2}\mathsf{x} + \frac{1}{2}\mathsf{x}' \in P_{t+1}$ so that equation (23) holds with equality. Then, by axiom A8 also equation (22) has to hold with equality. Equating the two equations by S'' yields the requirement

$$f_t^{-1}\Big\{\tfrac{1}{2} f_t\big[S + A\big] + \tfrac{1}{2} f_t\big[S' + A\big]\Big\} - A = f_{t+1}^{-1}\Big\{\tfrac{1}{2} f_{t+1}\big[S\big] + \tfrac{1}{2} f_{t+1}\big[S'\big]\Big\} . \tag{24}$$

for all $S, S' \in [\frac{1-\beta^{T-t}}{1-\beta}\, \underline{U}, \frac{1-\beta^{T-t}}{1-\beta}\, \overline{U}]$ and $A \in [\beta^{T-t}\, \underline{U}, \beta^{T-t}\, \overline{U}]$.

Part II: Observe that the right hand side of equation (24) is independent of A. Therefore, the left hand side has to be constant in A. The condition corresponds to a functional equation solved by Aczél (1966, 153). The only solutions for a continuous function f_t satisfying equation (24) are $f_t(z) = a_t \exp(\xi_t z) + b_t$ and $f_t(z) = a_t z + b_t$ with $a_t, b_t \in \mathbb{R}$ and $\xi_t \in \mathbb{R}, \xi_t \neq 0$ for $t \in \{1, ..., T-1\}$.

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[30] See proof of theorem 2 in Traeger (2007*b*), induction hypothesis 2.

Acknowledging that the left hand side of equation (24) is independent of A, both sides of equation (24) characterize an uncertainty aggregation rule with respect to a variation in S and S'. The parameterizing functions of uncertainty aggregation rules are unique up to affine transformations. As both sides have to coincide, f_{t+1} has to be an affine transformation of f_t. But affine transformations of the parameterizing function of an uncertainty aggregation rule do not affect the representation. Therefore, I can choose $f_t(z) = \exp(\xi z)$ for all $t \in \{1, ..., T\}$ in case of the first class of solutions. In the case that f_t is linear for one and thus for all t, I choose $f_t = \text{id}$ for all $t \in \{1, ..., T\}$. The latter case is identified with $\xi = 0$. In fact, observe that defining $\mathcal{M}^{\exp^0} = \lim_{\xi \to 0} \mathcal{M}^{\exp^\xi}$ yields by l'Hospital's rule:

$$\begin{aligned}\mathcal{M}^{\exp^0}(p_t, \tilde{u}_t) &\equiv \lim_{\xi \to 0} \mathcal{M}^{\exp^\xi}(p_t, \tilde{u}_t) = \lim_{\xi \to 0} \frac{\ln\left[\int dp_t \exp(\xi \tilde{u}_t)\right]}{\xi} \\ &= \lim_{\xi \to 0} \frac{\frac{\partial}{\partial \xi} \ln\left[\int dp_t \exp(\xi \tilde{u}_t)\right]}{\frac{\partial}{\partial \xi}\xi} = \lim_{\xi \to 0} \frac{\int dp_t \tilde{u}_t \exp(\xi \tilde{u}_t)}{\int dp_t \exp(\xi \tilde{u}_t)} \\ &= \frac{\int dp_t \tilde{u}_t}{1} = E_{p_t} \tilde{u}_t\,.\end{aligned}$$

Substituting the restricted uncertainty aggregation rules into the representation of theorem 1 yields the representation given in theorem 3.

Necessity: Part III: The representation is a special case of theorem 1. Therefore, axioms A1-A5 follow immediately from the necessity part of theorem 1. The following calculation shows that axiom A8 is satisfied for all $t \in \{1, ..., T-1\}$, $x \in X$ and $\mathrm{x}, \mathrm{x}', \mathrm{x}'' \in \mathrm{X}^{t+1}$ in the case $\xi \neq 0$:

$$\begin{aligned}&\tfrac{1}{2}(\mathrm{x}, x) + \tfrac{1}{2}(\mathrm{x}', x) \succeq_t (\mathrm{x}'', x) \\ \Leftrightarrow\ &\tfrac{1}{\xi} \ln\Big(\tfrac{1}{2} \exp\left[\xi \textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} u(\mathrm{x}_{\tau+1})\right] \exp\left[\xi \beta^T u(x)\right] \\ &\quad + \tfrac{1}{2} \exp\left[\xi \textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} u(\mathrm{x}'_{\tau+1})\right] \exp\left[\xi \beta^T u(x)\right]\Big) \geq \displaystyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} u(\mathrm{x}''_{\tau+1}) + \beta^T u(x) \\ \Leftrightarrow\ &\tfrac{1}{\xi} \ln\Big(\tfrac{1}{2} \exp\Big[\xi \textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} u(\mathrm{x}_\tau)\Big] + \tfrac{1}{2} \exp\Big[\xi \textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} u(\mathrm{x}'_\tau)\Big]\Big) \\ &\qquad \geq \textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} u(\mathrm{x}''_\tau)\end{aligned}$$

$$\Leftrightarrow \tfrac{1}{\xi}\ln\Big(\tfrac{1}{2}\exp\Big[\xi\tilde{u}_{t+1}(\mathsf{x})\Big]+\tfrac{1}{2}\exp\Big[\xi\tilde{u}_{t+1}(\mathsf{x}')\Big]\Big) \quad \geq \quad \tilde{u}_{t+1}(\mathsf{x}'')$$

$$\Leftrightarrow \tfrac{1}{2}\mathsf{x}+\tfrac{1}{2}\mathsf{x}' \succeq_{t+1} \mathsf{x}''\,.$$

In the case $\xi=0$ both sides of the above inequalities are linear. Thus, the term $\beta^T u(x)$ cancels as well and axiom A8 is satisfied.

Uniqueness: Part IV: The restriction that u is onto a given U^* fixes u uniquely, eliminating the affine freedom $\mathfrak{a}_0$ in the representation of theorem 1. Therefore, by the uniqueness result of theorem 1, the functions f_t are determined up to ('outer') affine transformations yielding unique measures

$$\mathrm{AIRA}^*_t(z) = -\frac{\frac{d^2}{dz^2}f_t(z)}{\frac{d}{dz}f_t(z)} = -\frac{\frac{d^2}{dz^2}a_t\exp(\xi z)+b_t}{\frac{d}{dz}a_t\exp(\xi z)+b_t} = -\xi$$

and accordingly

$$\mathrm{RIRA}^*_t(z) = -\frac{\frac{d^2}{dz^2}f_t(z)}{\frac{d}{dz}f_t(z)}z = -\xi z\,.$$

□

D Proofs for Section 6

Proof of theorem 4: The proof builds on a representation theorem derived in Traeger (2007*a*) for timing indifferent preferences in a non-stationary setting.

Sufficiency: By axioms A1-A5 and timing indifference axiom A9 theorem 4 in Traeger (2007*a*) brings about the following representation. For all $t\in\{1,...,T\}$ there exist strictly increasing continuous functions $u_t: X\rightarrow \mathrm{I\!R}$, such that defining the functions $\tilde{u}_t:\mathsf{X}^t\rightarrow\mathrm{I\!R}$ by

$$\tilde{u}_t(\mathsf{x}^t)=\textstyle\sum_{\tau=t}^{T}u_\tau(\mathsf{x}^t_\tau)$$

the expression $\mathcal{M}^{f_t}(p_t^{\mathsf{X}_t},\tilde{u}_t)$ represents preferences in the sense of equation (11). For a derivation of the lotteries $p^{\mathsf{X}_t}\in\Delta(\mathsf{X}^t)$ from their respective counterparts $p_t\in P_t=\Delta(\tilde{X}_t)$ by 'integrating out' the information on the timing of risk resolution I refer to Traeger (2007*a*). The identical reasoning as for the representation in theorem 1 shows that certainty stationarity in the sense of axiom 6 ensures the existence of a discount factor $\beta \in \mathbb{R}_{++}$ and a continuous function $u : X \to \mathbb{R}$ such that the functions u_t in the above representation can be chosen as $u_t = \beta^{t-1} u$.

As shown in the proof of theorem 4 in Traeger (2007*a*) the functions f_t in the recursive representation differ from the parameterizing function of the uncertainty aggregation rules in the non-recursive representation by a normalization factor which, in the stationary setting, coincides with the discount factor. Therefore, the representation in theorem 4 corresponds to a representation in the sense of theorem 1 with the choice $f_t(z) = \exp(\xi \beta^{t-1} z)$ as is easily verified by the following calculation:

$$\begin{aligned}
\mathcal{M}^{f_t}(p_t, \tilde{u}_t) &= f_t^{-1}\Big(\mathrm{E}_{p_t} f_t \Big[u(x_t) + \beta\, f_{t+1}^{-1}\Big(\mathrm{E}_{p_{t+1}} f_{t+1} \circ \tilde{u}_{t+1} \Big) \Big] \Big) \\
&= \tfrac{1}{\beta^{t-1}\xi} \ln \Big(\mathrm{E}_{p_t} \ \exp \Big[\beta^{t-1}\, \xi \, \Big\{ u(x_t) + \beta \, \tfrac{1}{\beta^t \xi} \ln \Big(\mathrm{E}_{p_{t+1}} \ \exp \big[\beta^t \, \xi \ \tilde{u}_{t+1} \big] \Big) \Big\} \Big] \Big) \\
&= \tfrac{1}{\beta^{t-1}\xi} \ln \Big(\mathrm{E}_{p_t} \ \exp \Big[\beta^{t-1}\, \xi \ u(x_t) \Big] \mathrm{E}_{p_{t+1}} \ \exp \big[\beta^t \, \xi \ \big(u(x_t) + \\
&\qquad\qquad\qquad\qquad \beta\, \mathcal{M}^{f_{t+2}}(p_{t+2}, \tilde{u}_{t+2}) \big) \big] \Big) \\
&= \tfrac{1}{\beta^{t-1}\xi} \ln \Big(\mathrm{E}_{p_t} \ \exp \big[\beta^{t-1}\, \xi \ u(x_t) \big] \ \mathrm{E}_{p_{t+1}} \ \exp \big[\beta^t \, \xi \ u(x_t) \big] \ \mathrm{E}_{p_{t+1}} ... \Big)
\end{aligned}$$

which omitting the details on integrating out the information on the timing of risk resolution becomes

$$\begin{aligned}
\mathcal{M}^{f_t}(p_t, \tilde{u}_t) &= \tfrac{1}{\beta^{t-1}\xi} \ln \Big(\mathrm{E}_{p_t^{\mathcal{X}}} \ \exp \big[\beta^{t-1}\, \xi \ u(x_t) \big] \ \exp \big[\beta^t \, \xi \ u(x_t) \big] ... \Big) \\
&= \tfrac{1}{\beta^{t-1}\xi} \ln \Big(\mathrm{E}_{p_t^{\mathcal{X}}} \ \exp \big[\xi \ \textstyle\sum_{\tau=t}^{T} \beta^{\tau-1} \ u(x_\tau) \big] \Big)
\end{aligned}$$

which is a strictly increasing transformation of the expression in the representing equation (11). Therefore, the measures of intertemporal risk aversion are immediately calculated from the relation $f_t(z) = \exp(\xi \beta^{t-1} z)$, yielding the results stated in the theorem.

Necessity: Necessity of the axioms follows from their necessity in the representations of 1 above and theorem 4 in Traeger (2007*a*).

Uniqueness: Uniqueness of u and the measures of intertemporal risk aversion is an immediate consequence of proposition 1

□

E Proofs for Section 7

Proof of theorem 5: The assertion follows from a comparison of the time evolvement of intertemporal risk aversion in the representation of theorem 3 and theorem 4.
Sufficiency: The resulting representation has to be a special case of the representation derived in theorem 1 that satisfies risk stationarity A8 as well as timing indifference A9. I have shown in the proof of theorem 3 that risk stationarity implies that the representing functions f_t characterizing uncertainty aggregation in the representation of theorem 1 have to be coincide for different periods (up to possibly time dependent affine transformations that do not affect uncertainty aggregation). In contrast I have shown in the proof of theorem 4 that timing indifference implies that the functions f_t describing uncertainty aggregation in the representation of theorem 1 have to be of the form $f_t(z) = \operatorname{sgn}(\xi) \exp(\xi \beta^{t-1} z)$ for all $t \in \{1, ..., T\}$. Moreover, for a strictly intertemporal risk averse decision maker theorem 2 a) implies $\xi < 0$.[31] For $\beta \neq 1$ the implied change of f_t over time contradicts the constancy required by axiom A8 (there is no affine transformation taking f_t into f_t' for $t \neq t'$ if it is of the given form). Thus the only representation that satisfies both axioms has to have a discount rate $\beta = 1$.
Necessity: Necessity follows from necessity in theorems 3 and 4 and the fact that both representation coincide for $\beta = 1$. □

Proof of proposition 2: The cumulative probability distribution of welfare in period t is defined as $\mathrm{P}_t(z) = \mathrm{p}_t^u([0, z])$ on the interval $[0, 1]$ and continuous from the right. By theorem 2 in Rothschild & Stiglitz (1970, 237) equations (14) and (16) imply that for any strictly increasing concave function $f \in \mathcal{C}^0([0, 1])$ it holds

$$\begin{aligned} & \mathrm{E}_{\mathrm{p}_t^u} f && \geq \mathrm{E}_{\mathrm{p}_{t+1}^u} f \\ \Rightarrow\ & f^{-1}\big(\mathrm{E}_{\mathrm{p}_t^u} f\big) && \geq f^{-1}\big(\mathrm{E}_{\mathrm{p}_{t+1}^u} f\big) \,. \end{aligned}$$

The uncertainty aggregation rule $\mathcal{M}^{\exp^\xi}(\mathrm{p}_t, u)$ is characterized by the strictly increasing and concave function $f(z) = -\exp(\xi\, z)$.[32] Therefore it follows from $f^{-1}\big(\mathrm{E}_{\mathrm{p}_t^u} f\big) = f^{-1}\big(\mathrm{E}_{\mathrm{p}_t} f \circ u\big) = \mathcal{M}^f(\mathrm{p}_t, u)$ that equation (17) is satisfied as a weak inequality. The

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[31] Because $f_t^*(z) = \exp(\xi \beta^{t-1} z)$ is decreasing in z the theorem has to be applied to the function $f_t = \operatorname{sgn}(\xi) \exp(\xi \beta^{t-1} z)$ which increases and characterizes the same uncertainty aggregation rule.

[32] Which describes the same uncertainty aggregation rule as the decreasing function $\exp^\xi$.

strictness of the inequality follows from equation (15) and the fact that the distribution functions P_t are continuous from the right. □

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