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CESifo Working Papers

ON ADJUSTING THE HP-FILTER FOR THE FREQUENCY OF OBSERVATIONS*

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CESifo Working Paper No. 479

May 2001

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* We are grateful to James Stock, Albert Marcet, and Dan Knudsen for useful comments. We also thank Dave Backus for provision of data.

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Abstract

This paper studies how the HP-Filter should be adjusted, when changing the frequency of observations. It complements the results of Baxter and King (1999) with an analytical analysis, demonstrating that the filter parameter should be adjusted by multiplying it with the fourth power of the observations frequency ratios. This yields an HP parameter value of 6.25 for annual data given a value of 1600 for quarterly data. The relevance of the suggestion is illustrated empirically.

JEL Classification: C32, E32.

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1 Introduction

The Hodrick and Prescott (1980, 1997) filter (the HP-filter hereafter) has become a standard method for removing trend movements in the business cycle literature. The filter has been applied both to actual data (see e.g. Backus and Kehoe, 1992, Blackburn and Ravn, 1992, Brandner and Neusser, 1992, Danthine and Girardin, 1989, Danthine and Donaldson, 1993, Fiorito and Kollintzas, 1994, and Kydland and Prescott, 1990) and in studies where artificial data from a model are compared with the actual data (see e.g. Backus, Kehoe and Kydland, 1992, Cooley and Hansen, 1989, Hansen, 1985, and Kydland and Prescott, 1982).

Although the use of the HP-filter has been subject to heavy criticism (see e.g. Canova, 1994, 1998, Cogley and Nason, 1995, Harvey and Jaeger, 1993, King and Rebelo, 1992, or Söderlind, 1994), it has withstood the test of time and the fire of discussion remarkably well. Thus, although elegant new band-pass filters are being developed (see Baxter and King, 1999, Baxter, 1994, and Christiano and Fitzgerald, 1999), it is likely that the HP-filter will remain one of the standard methods for detrending.

Most applications of this filter have been to quarterly data but data is often only available at the annual frequency while in other cases monthly data might be published. This raises the question of how one can adjust the HP-filter to the frequency of the observations so that the main properties of the results are conserved across alternative sampling frequencies. While most researchers have followed Hodrick and Prescott (1980, 1998) and used the value of 1600 for the smoothing parameter when using quarterly data, there is less agreement in the literature when moving to other frequencies. Backus and Kehoe (1992) use a value of 100 for annual data while Correia, Neves and Rebelo (1992) and Cooley and Ohanian (1991) suggest a value of 400.

Baxter and King (1999) have recently shown that a value of around 10 for annual data is much more reasonable. They arrive at this value by visually inspecting the transfer function of the HP-filter for annual data and comparing it to a bandpass filter. A similar value has already been obtained earlier by Hassler et al (1992) by investigating the average cycle length obtained in a time series of output.

This paper complements these insights, using two different analytical approaches. The first approach uses the time domain and focuses on the ratio of the variance of the cyclical component to the variance of the second difference of the trend component: this ratio is often used for calculating the smoothing parameter. For a particular benchmark stochastic process, it is shown that time-aggregation changes this ratio by the fourth power of the observation frequency. The second approach uses the frequency domain and investigates the transfer function of the HP filter, thereby obtaining a general result. Again, a change-of-variable argument shows that one should adjust the HP parameter with approximately the fourth power of the frequency change. Both approaches therefore yield a value of around $1600/4^4 = 6.25$ for annual data, which is close to the value of 10 given in Baxter and King (1999).

We then show that our recommendations work extremely well on US GDP data: Using a value of the smoothing parameter of 6.25 for annual data and 1600 for quarterly data produces almost exactly the same trend. This leads us to reconsider the business cycle "facts" reported in earlier studies. As an example, we cast doubt on a finding by Backus and Kehoe (1992) on the historical changes in output volatility and return instead to older conventional wisdom (Baily, 1978, Lucas, 1977): output volatility turns out to have decreased after the Second World War.

The remainder of the paper is organized as follows. Section 2 presents the HP-filter and provides the first, time-domain-based approach, whereas section 3 provides the second, frequency-domain-based approach. In Section 4, we recompute some facts about business cycles. Finally Section 5 concludes.

2 A time-domain perspective

The HP-filter removes a smooth trend τ_t from some given data y_t by solving

$$\min_{\tau_t} \sum_{t=1}^{T} \left((y_t - \tau_t)^2 + \lambda \left((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right)^2 \right)$$

The residual $y_t - \tau_t$ (the deviation from the trend) is then commonly referred to as the "business cycle component".

The filter involves the smoothing parameter λ which penalizes the acceleration in the trend relative to the business cycle component. Researchers typically set $\lambda = 1600$ when working with quarterly data. However, data does not always come at quarterly intervals. It may even be desirable to move to annual, monthly or some other time interval of observation instead.

Thus, the question arises how the HP filter should be adjusted for the frequency of observations. This question is the focus of this paper. We do not investigate whether the HP filter is desirable per se or aim at a comparison to some optimal band pass filter as in Baxter and King (1999). Rather we take it as granted that a researcher wishes to filter the data using the HP filter, and ask how the parameter λ should be adjusted when changing the sampling frequency.

A popular perspective on the smoothing parameter in the literature is to consider the decomposition of some given time series y_t into a trend τ_t and a cycle c_t,

$$y_t = \tau_t + c_t \tag{1}$$

If c_t as well as the second difference of τ_t are normally and independently distributed, then the HP filter is known to be optimal, and λ is given as the ratio of the two variances, $\lambda = \sigma_c^2/\sigma_{\Delta^2\tau_t}^2$ (see Hodrick and Prescott, 1980, 1997, or King and Rebelo, 1993). Generally, the HP filter is not optimal for estimating this decomposition. In particular, even if the HP filter is optimal for (1), it is unlikely to be optimal when time-aggregating the process (1) because time-aggregation usually introduces moving average terms. As our focus is on adjusting λ, when changing the frequency of observation, we shall however ignore the issue of optimal filtering, and instead simply focus on the question of how the ratio of the variances change. Below we look at the adjustment of the smoothing parameter using a continuous time approach. In the appendix we show how one can derive similar results using a discrete time approach and we relate the results more directly to the theory of optimal filtering.

It is convenient to consider a benchmark continuous-time version of (1) which satisfies the conditions stated above, i.e. where the cycle as well as the second difference of the trend are independently and normally distributed, taking the form of Brownian motion increments. We then analyze the change in the variances when observing the process at discrete time intervals. Let y_t be the "flow" dz_t of some stochastic process z_t with

$$dz_t = \tau_t dt + \sigma_c dW_t^1 \tag{2}$$

where

$$d\tau_t = \mu_t dt, \;\; d\mu_t = \sigma_\tau dW_t^2 \tag{3}$$

and dW_t^1 and dW_t^2 are two independent Brownian motions. There are two possibilities for observing the process at some discrete time interval α, say: these observations may be time aggregated (or time averaged) or they may be sampled at these discrete time intervals, see Christiano and Eichenbaum (1986).

Consider time aggregation first, i.e. for some length $\alpha > 0$, consider observing

$$y_{t;\alpha} = \int_{s=0}^{\alpha} dz_{t-s} = \tau_{t;\alpha} + c_{t;\alpha}$$

where

$$\begin{aligned} \tau_{t;\alpha} &= \int_{s=0}^{\alpha} \mu_{t-s} ds \\ c_{t;\alpha} &= \int_{s=0}^{\alpha} \sigma_c dW_t^1 \end{aligned}$$

For any stochastic process x_t, define the α-differencing operator

$$\Delta_\alpha x_t = x_t - x_{t-\alpha}$$

We are interested in how

$$\lambda_\alpha = \frac{\sigma^2(c_{t;\alpha})}{\sigma^2\left(\Delta_\alpha^2 \tau_{t;\alpha}\right)}$$

changes with α.[1]

Clearly,

$$\sigma^2(c_{t;\alpha}) = \alpha\sigma_c^2 = \alpha\sigma^2(c_{t;1})$$

For $\Delta_\alpha^2 \tau_{t;\alpha}$, introduce first $x_t = \Delta_\alpha \tau_{t;\alpha}$ and write it as

$$\begin{aligned} x_t &= \int_{s_1=0}^{\alpha} \left(\mu_{t-s_1} - \mu_{t-\alpha-s_1}\right) ds_1 \\ &= \int_{s_1=0}^{\alpha} \int_{s_2=0}^{\alpha} d\mu_{t-s_1-s_2} ds_1 \end{aligned}$$

Substitute $d\mu_{t-s_1-s_2} = x_{t-s_1-s_2} ds_2$ and repeat this calculation to obtain an expression of the second α-difference,

$$\begin{aligned} \Delta_\alpha^2 \tau_{t;\alpha} &= \sigma_\tau \int_{s_1=0}^{\alpha} \int_{s_2=0}^{\alpha} \int_{s_3=0}^{\alpha} dW^2_{t-s_1-s_2-s_3} ds_2 ds_1 \\ &= \sigma_\tau \int_{s=0}^{3\alpha} A(s;\alpha) dW^2_{t-s} \end{aligned}$$

where

$$A(s;\alpha) = \int_{s_1=0}^{\alpha} \int_{s_2=0}^{\alpha} 1_{[0,\alpha]}(s - s_1 - s_2) ds_2 ds_1$$

and where the last equality was obtained by a change of variables, $s = s_1 + s_2 + s_3$. The variance is therefore given by

$$\sigma^2\left(\Delta_\alpha^2 \tau_{t;\alpha}\right) = \sigma_\tau \int_{s=0}^{3\alpha} A(s;\alpha)^2 ds \tag{4}$$

While one could calculate $A(s;\alpha)$, one does not have to. Simply observe that

$$A(s;\alpha) = \alpha^2 A(s/\alpha; 1)$$

With one more change of variable to $\tilde{s} = s/\alpha$ in (4), we finally find

$$\sigma^2\left(\Delta_\alpha^2 \tau_{t;\alpha}\right) = \alpha^5 \sigma_\tau \int_{\tilde{s}=0}^{3} A(\tilde{s};1)^2 d\tilde{s} = \alpha^5 \sigma^2\left(\Delta_1^2 \tau_{t;1}\right)$$

and hence

$$\lambda_\alpha = \frac{1}{\alpha^4}\lambda_1$$

[1] One can equally well divide the processes by α to obtain time averaging rather than time aggregation: this makes no difference for λ_α and the calculation is very similar.

i.e., the HP parameter λ should be adjusted with the fourth power of the frequency change. This finding will be reconfirmed in section 3, using another approach.

For sampling at discrete time intervals α the calculations become simpler yet. Suppose we observe[2] the flow $y_t = dz_t$ at intervals α. The diffusion part still has variance $\sigma_c^2 dt$. What needs to be calculated is the variance of $\Delta_\alpha^2 \tau_t$. The same calculation as above leads to

$$\begin{aligned} \Delta_\alpha^2 \tau_t &= \int_{s_1=0}^{\alpha} \int_{s_2}^{\alpha} \sigma_\tau dW_{t-s_1-s_2}^2 \\ &= \int_{s=0}^{2\alpha} B(s;\alpha) dW_{t-s} \end{aligned}$$

where

$$B(s;\alpha) = \int_{s_1=0}^{\alpha} 1_{[0,\alpha]}(s-s_1) ds_1 = \alpha B(s/\alpha;1)$$

Similar to the calculation above,

$$\lambda_\alpha^{(s)} = \frac{\sigma_c^2 dt}{\sigma^2(\Delta_\alpha^2 \tau_t)} = \frac{1}{\alpha^3} \lambda_1^{(s)}$$

i.e., the smoothing parameter for the HP filter should be adjusted using the third power of α. This result differs from the fourth-power result for time-averaged data above, but also differs from the literature suggestion of adjusting with the second or the first power of α.

In practice, one may therefore wonder whether adjustment with the fourth or the third power is more appropriate. Our recommendation here is to always use the fourth power rather than the third. First, most macroeconomic time series are time-averaged, so that the calculation above would suggest adjusting with the fourth power anyhow. But even for the sampling case, simulations of the process above shows that adjusting with the fourth power rather than the third produces essentially the same trend. The next section can be read as an explanation, why this is the case.

3 A Frequency-Domain Perspective

An alternative way to look at the issue is from a frequency domain perspective. This allows us to provide a general result, as we no longer need to assume the special structure (2,3). The transfer function of the HP-filter is given by (see e.g. King and Rebelo, 1993)

$$h(\omega;\lambda) = \frac{4\lambda(1-\cos(\omega))^2}{1+4\lambda(1-\cos(\omega))^2} \tag{5}$$

[2]"Observing" should be understood here in the sense that the continous-time limit approximates some discrete-time process at very small time intervals.

This filter is similar to a high-pass filter (see e.g. Ravn and Uhlig, 1997, or Baxter and King, 1999, for a plot of the transfer function). Choosing different values for λ is comparable to choosing different values for the cut off point of the high pass filter.

Let $h(\omega; \lambda_1)$ be the filter representation for quarterly data and let $h(\omega/s; \lambda_s)$ be the filter representation for an alternative sampling frequency s, where we let s be the ratio of the frequency of observation compared to quarterly data ($s = 1/4$ for annual data or $s = 3$ for monthly data). Then, ideally, we would like to have:

$$h(\omega; \lambda_1) \approx h(\omega/s; \lambda_s) \tag{6}$$

While this cannot hold exactly for all ω, it should hold at least approximately[3]. In order to derive the appropriate adjustment rule λ_s one could, in principle, find λ_s as to minimize some distance metric between $h(\omega; \lambda_1)$ and $h(\omega/s; \lambda_s)$. However, we take a short-cut to this and specify a simple functional rule for this adjustment process: We apply the simple criterion to multiply λ with some power of the frequency adjustment, i.e. to choose

$$\lambda_s = s^m \lambda_1 \tag{7}$$

Thus, the problem is to choose m so as to fit (6).

Consider a marginal change in the observation frequency ratio s around $s = 1$, and look at its differential impact on the HP-filter. For the correct adjustment, is should be the case that

$$\frac{d}{ds} h(\omega/s; \lambda_s) \approx 0 \tag{8}$$

where $\frac{d}{ds}$ denotes the total derivative with respect to s. For each ω and s, this equation can be solved for the parameter $m = m(s, \omega)$: one finds:

$$m(s, \omega) = 2 \frac{\omega/s \sin(\omega/s)}{1 - \cos(\omega/s)} \tag{9}$$

If the power specification is appropriate, then this expression should be approximately constant over the range of "relevant" frequencies ω. Inspection of the transfer function shows that it suffices to restrict attention to values $0 \leq \omega \leq \pi/5$, see Ravn and Uhlig, 1997. Table 1 lists values of $m = m(1, \omega) = m(s, \omega s)$ for ω in this range. The values in this table suggest that $m = 4$ or something close to it is an excellent choice if one wishes to make the transfer function invariant to the frequency of observation, thereby reconfirming the results of section 2 for time-aggregated data. The analysis furthermore shows that $m = 4$ is the

[3] By this equation we do not mean to say that the HP-filter is "optimal" in any sense; Rather it says that as the frequency of the observations is altered, the filter - being optimal or not - should have approximately the same properties.

exact outcome only at $\omega = 0$: otherwise, a slightly lower number between, say, $m = 3.8$ and $m = 4$ might be more appropriate.

Thus, for $\lambda_{quarterly} = 1600$, this implies that $\lambda_{annual} = 1600/4^4 = 6.25$ (or 8.25 for $m = 3.8$) and $\lambda_{monthly} = 1600 \cdot 3^4 = 129600$ (104035 for $m = 3.8$).

Given these results, we now check how well this adjustment rule works in practise. We examine US real GDP from the Bureau of Economic Analysis for the period 1947-2000 sampled at the quarterly and the annual frequency. We compare the trend component of the quarterly data using $\lambda_{quarterly} = 1600$ with the trend components of the annual data using $\lambda_{annual} = 400, 100, 25$, or 6.25. The results are shown in Figure 1.[4] This picture clinches our case once more: the trend component of the quarterly data using $\lambda_{quarterly} = 1600$ and the trend component of the annual data using $\lambda_{annual} = 6.25$ are practically identical whereas large differences are visible for $\lambda_{annual} = 400, 100$ or 25.

4 Recomputing the Facts.

Based on the above analysis it seems natural to ask whether the modification of the rule for adjusting the smoothing parameter matters for reported business cycle "facts". As an application we recompute of some of the results reported by Backus and Kehoe (1992) for a cross-section of OECD countries using historical annual data: these authors have used $\lambda_{annual} = 100$ while we shall use $\lambda_{annual} = 6.25$.

One of Backus and Kehoe's (1992) most interesting findings was that output volatility was higher in the interwar period than during the postwar period but that there is no general rule as far as a comparison of the postwar period with the prewar (pre WWI) period is concerned. This result is in contrast to the conventional wisdom of e.g. Burns (1960), Lucas (1977), and Tobin (1980) that output volatility declined after WWII relative to both earlier periods. Another interesting result was that prices changed from generally being procyclical before World War II to being countercyclical thereafter.

Table 3 lists the results for output volatility when using our recommended value for the smoothing parameter. We find, that the difference in volatility between the prewar and the postwar period generally narrows, and that for most countries, there has been a decline in volatility in the postwar period relative to either the interwar period or the prewar period.[5] In contrast to Backus and Kehoe (1992) these results are in line with the traditional

[4] In order to make the results visually clearer, we have removed a linear trend from the HP-filter trend components.

[5] By this we do not mean to challenge Romer's (1989) argument that the high prewar volatility is due to measurement error. One should however notice that e.g. UK data do not suffer from these measurement

wisdom quoted above. This is an important result that Baily (1978) and Tobin (1980) have interpreted in terms of stabilization policy.

Table 4 reports the results for the cyclical behavior of the price level. There, and except for Norway, our results reconfirm the finding by Backus and Kehoe (1992), that prices have become countercyclical in the postwar period and that the interwar period

historically was the period where procyclicality was most pronounced. I.e., this result seems to be fairly robust to the choice of the smoothing parameter. These results are also in line with other studies, see e.g. Cooley and Ohanian (1991) and Ravn and Sola (1995).

5 Conclusions

This paper provides an analytic investigation into how the smoothing parameter λ of the HP filter should be adjusted, when changing the frequency of observation. The major conclusion is that the parameter λ should be adjusted according to the fourth power of a change in the frequency of observations. For annual observations, this suggest to set $\lambda = 6.25$, which is close to the value found in Baxter and King (1999), but different from the value $\lambda = 100$ or $\lambda = 400$ typically found in the literature. Some well-known comparisons of business cycles moments across countries and time periods have been recomputed, using the recommended fourth-power adjustment. In particular, we cast doubt on a finding by Backus and Kehoe (1992) and return instead to older conventional wisdom (Baily, 1978, Lucas, 1977, Tobin, 1980): based on the new HP-Filter adjustment rule, output volatility turns out to be lower in the postwar period compared to the prewar period.

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6 Appendix: Temporal Aggregation: An Optimal Filter Approach

This appendix describes the link between optimal filters and the adjustment of the smoothing parameter in the HP-filter to the frequency of observations. Here we use a discrete time approach and the appendix extends Section 2 to this case and shows the temporal aggregation problem's properties in more detail.

In order to make the connection between temporal aggregation and the choice of the smoothing parameter in the HP filter, we need to relate the HP-filter to the properties of the time-series under consideration. A particular helpful way of making this connection is to look at "optimal filters".

Suppose that a time-series under consideration can be written as:

$$y_t = \tau_t + c_t$$

where c_t is the "cyclical component". Suppose also that these two components are generated by ARIMA processes:

$$\begin{aligned} A_\tau (L)\, \tau_t &= B_\tau(L) e_t \\ A_c (L)\, c_t &= B_c(L) \varepsilon_t \end{aligned}$$

where L is the lag operator (defined by $L^s x_t = x_{t-s}$), and $e_t \sim nid\,(0, \sigma_\tau^2)$, $\varepsilon_t \sim nid\,(0, \sigma_c^2)$, and e_t and ε_t are orthogonal. Whittle (1963) have shown that the optimal signal extraction filter for the trend component (defined by $R\,(L)\, y_t = \tau_t$) is given by:

$$R\,(L) = \frac{F\,(L)}{D(L) + F\,(L)}$$

where

$$\begin{aligned} D(L) &= \frac{B_c\,(L)\, B_c\,(L^{-1})}{A_c\,(L)\, A_c\,(L^{-1})} \sigma_c^2 \\ F\,(L) &= \frac{B_\tau\,(L)\, B_\tau\,(L^{-1})}{A_\tau\,(L)\, A_\tau\,(L^{-1})} \sigma_\tau^2 \end{aligned}$$

The trend component for the HP-filter can be written as:

$$R^{HP}\,(L) = \frac{1}{1 + \lambda\,(1-L)^2\,(1-L^{-1})^2}$$

As highlighted by Hodrick and Prescott (1980) and King and Rebelo (1993) sufficient condition for optimality of the HP-filter are: (a) $A_\tau(L) = (1-L)^2$, (b) $B_\tau(L) = A_c(L) = B_c(L) = 1$, and (c) $\lambda = \sigma_c^2/\sigma_\tau^2$.

Let $y_t^{F,N}$ denote the N-period sum of y_t ("average sampled" in Christiano and Eichenbaum's, 1986, terminology). Similarly let $y_t^{S,N}$ denote a point-in-time sampled variable (so that y_t is sampled at periods $1, 1+N, 1+2N, ...$).

$y_t^{F,N}$ is defined as:

$$y_t^{F,N} = \sum_{s=0}^{N-1} y_t = \sum_{s=0}^{N-1} (\tau_t + c_t) = \tau_t^{F,N} + c_t^{F,N} \tag{10}$$

where:

$$\tau_t^{F,N} = \sum_{s=0}^{N-1} \tau_s = \left(\sum_{s=0}^{N-1} L^s\right) \tau_t \tag{11}$$

$$c_t^{F,N} = \sum_{s=0}^{N-1} c_s = \left(\sum_{s=0}^{N-1} L^s\right) c_t \tag{12}$$

For a variable that is point-in-time sampled, we have that:

$$y_t^{S,N} = \tau_t^{S,N} + c_t^{S,N} \tag{13}$$

$$\tau_t^{S,N} = \tau_t \tag{14}$$

$$c_t^{S,N} = c_t \tag{15}$$

Since the intervals over which the series are temporally aggregated are non-overlapping, the cyclical components will be independently distributed over time and $c_t^{F,N} \sim nid\,(0, N\sigma_c^2)$, $c_t^{S,N} \sim nid\,(0, \sigma_c^2)$.

Thus, we can concentrate on the properties of the second differences of the "trend" components given by:

$$\triangle_N^2 \tau_t^{F,N} = (1-L)^2 \left(\sum_{s=0}^{N-1} L^s\right)^3 \tau_t = (1-L)^2 H_N^F(L)\, \tau_t$$

$$\triangle_N^2 \tau_t^{S,N} = (1-L)^2 \left(\sum_{s=0}^{N-1} L^s\right)^2 \tau_t = (1-L)^2 H_N^S(L)\, \tau_t$$

where:

$$H_N^F(L) = \left(\sum_{s=0}^{N-1} L^s\right)^3 = \sum_{s=0}^{3(N-1)} h_N^F(s)\, L^s \tag{16}$$

$$\begin{aligned}
h_N^F(s) &= N(s+1-\max(s-N+1,0)) \\
&\quad - \sum_{j=\max(s-N+1,0)}^{s} \min(N,|N-1-j|) \qquad (17) \\
H_N^S(L) &= \left(\sum_{s=0}^{N-1} L^s\right)^2 = \sum_{s=0}^{2(N-1)} h_N^S(s)\, L^s \\
h_N^S(s) &= N-\min(N,|N-s-1|) \qquad (18)
\end{aligned}$$

The second difference of the trend component has an MA(2) structure for the average sampled data and an MA(1) structure for the point-in-time sampled data (see Christiano and Eichenbaum, 1986, for a general continuous time treatment of this issue[6]). Thus, $\triangle_N^2 \tau_t^{F,N}$ and $\triangle_N^2 \tau_t^{S,N}$ are (unconditionally) mean zero normally distributed variables. The variances of these second differences are given by[7]:

$$\begin{aligned}
\sigma_{F,N}^2 &= E\left(\triangle_N^2 \tau_t^{F,N}\right)^2 = D_N^F \sigma_\tau^2,\ D_N^F \equiv \left(\sum_{i=0}^{3(N-1)} h_N^F(s)^2\right) \\
\sigma_{S,N}^2 &= E\left(\triangle_N^2 \tau_t^{S,N}\right)^2 = D_N^S \sigma_\tau^2,\ D_N^S \equiv \left(\sum_{i=0}^{2(N-1)} h_N^S(s)^2\right) = \frac{2m^3+m}{3}
\end{aligned}$$

and the autocovariances are given by:

$$\begin{aligned}
E\left(\triangle_N^2 \tau_t^{F,N} \triangle_N^2 \tau_{t-iN}^{F,N}\right) &= \left(\sum_{s=iN}^{3(N-1)} h_N^F(s)\, h_N^F(s-iN)\right)\sigma_\tau^2 \\
E\left(\triangle_N^2 \tau_t^{S,N} \triangle_N^2 \tau_{t-iN}^{S,N}\right) &= \left(\sum_{s=iN}^{2(N-1)} h_N^S(s)\, h_N^S(s-iN)\right)\sigma_\tau^2 = \frac{m^2-1}{2(2m^2+1)}\sigma_\tau^2
\end{aligned}$$

where the analytical results again was derived by Working (1960). It is possible to show that:

$$\begin{aligned}
\lim_{N\to\infty} E\left(\triangle_N^2 \tau_t^{F,N} \triangle_N^2 \tau_{t-N}^{F,N}\right)/\sigma_\tau^2 &\approx 0.40 \\
\lim_{N\to\infty} E\left(\triangle_N^2 \tau_t^{F,N} \triangle_N^2 \tau_{t-2N}^{F,N}\right)/\sigma_\tau^2 &\approx 0.015 \\
\lim_{N\to\infty} E\left(\triangle_N^2 \tau_t^{S,N} \triangle_N^2 \tau_{t-N}^{F,N}\right)/\sigma_\tau^2 &= 0.25
\end{aligned}$$

[6] Their results do not carry over directly to our case because one of their auxiliary assumptions (a rational representation of the integral of continous time first difference function) is violated in our example.

[7] The analytical result for $\sigma_{S,N}^2$ was derived originally by Working (1960) for the temporal aggregation of a random walk that "average sampled". When the first difference of the variable is a random walk, this result applies to the point-in-time temporal aggregation.

This implies that the HP filter no longer is "optimal" filter for the temporally aggregated data. The optimal filter will now be given by:

$$
\begin{aligned}
R^{F,N}(L) &= \frac{B_{\tau}^{F,N}(L)\,B_{\tau}^{F,N}(L^{-1})}{B_{\tau}^{F,N}(L)\,B_{\tau}^{F,N}(L^{-1})+\lambda(1-L)^2(1-L^{-1})^2} \\
R^{S,N}(L) &= \frac{B_{\tau}^{S,N}(L)\,B_{\tau}^{S,N}(L^{-1})}{B_{\tau}^{S,N}(L)\,B_{\tau}^{S,N}(L^{-1})+\lambda(1-L)^2(1-L^{-1})^2}
\end{aligned}
$$

where $B_{\tau}^{F,N}(L)$ is a lag polynomial of order 2, and $B_{\tau}^{S,N}(L)$ is a lag polynomial of order 1. Nevertheless, we can still provide some insights from this approach. Since the temporally aggregated data are serially correlated, the optimal filter will associate a larger proportion of the movements in the data with movements in the trend. Thus, since the HP-filter will associate more movements in the data to the trend the lower is λ, ignoring the serial correlation when adjusting λ can be though of as providing an upper bound for this parameter when moving from high frequency data to low frequency data.

Table 1 lists the results from adjustments of the smoothing parameter ignoring the problem of serial correlation. In this table N denotes the frequency at which the data is generated (i.e. $N = 4$, for example, the frequency is a quarter). For each N we then compute the values of variance of the cyclical component and the second difference of the trend component at the monthly, quarterly and annual frequency. We then assume that $\lambda_{quarterly}$ (the value of the smoothing parameter at the quarterly frequency) is equal to 1600 and given this we can compute $\lambda_{monthly}$ (for $N \geq 12$) and λ_{annual}. These values will correspond to "upper bounds" for the annual data (as just explained) and "lower bounds" for the monthly data.

Thus, for average sampled data we obtain (for $N \geq 4$) a value of λ_{annual} in the neighborhood of 6-7 if $\lambda_{quarterly} = 1600$. For point-in-time aggregated variables, we find that the implied value of the smoothing parameter at the annual frequency is close to 25.

These values differ significantly from $\lambda_{annal} = 100$ or 400, previously applied in the literature but are close to the values suggested by Hassler et al (1992) and Baxter and King (1999) for the case of average sampled data and N sufficiently large. Our analytical argument complements both of these studies and provides a more general insight into the issue.

At the monthly frequency, the appropriate value of the smoothing parameter appears to be quite sensitive to the frequency at which the data is generated. However, for large "enough" values of N, the implied value of the smoothing parameter is around 130000 for averaged data and 44000 for point-in-time sampled data.

ω	0	$\pi/20$	$\pi/10$	$\pi/5$
$m(1,\omega)$	4	3.992	3.967	3.868

Table 1: *The optimal power adjustment at frequency ω for an adjustment locally around a quarterly sampling rate. As one can see, the optimal adjustment is generally between* 3.8 *and* 4.0 *at the relevant frequencies.*

	Standard Deviations (%)			$n = 4$		$n = 2^*$	
	I.Prewar	II.Interwar	III.Postwar	I/III	II/III	I/III	II/III
Australia	3.77(0.37)	2.47(0.35)	1.40(0.14)	2.69	1.77	3.3	2.5
Canada	3.13(0.27)	5.06(0.77)	1.50(0.21)	2.09	3.38	2.0	4.4
Denmark	2.20(0.17)	2.45(0.37)	1.35(0.15)	1.63	1.82	1.6	1.8
Germany	2.32(0.21)	5.26(0.88)	1.80(0.24)	1.29	2.92	1.5	4.4
Italy	2.13(0.20)	2.60(0.30)	1.51(0.14)	1.41	1.72	1.2	1.8
Japan	2.10(0.27)	2.47(0.38)	1.45(0.18)	1.45	1.70	0.8	1.0
Norway	1.07(0.09)	2.89(0.56)	1.06(0.12)	1.01	2.72	1.1	2.0
Sweden	1.73(0.22)	2.41(0.47)	1.03(0.09)	1.68	2.34	1.7	2.6
United Kingdom	1.54(0.16)	2.50(0.30)	1.27(0.17)	1.21	1.97	1.3	2.1
United States	3.30(0.35)	4.91(0.70)	1.58(0.17)	2.09	3.11	1.9	4.1

Table 2: **Output Volatility.** **Numbers from Backus and Kehoe (1992). Numbers in parentheses are standard errors computed from GMM estimations of the unconditional moments.*

	$n = 4$			$n = 2^*$		
	I.Prewar	II.Interwar	III.Postwar	I.Prewar	II.Interwar	III.Postwar
Australia	0.29	0.30	-0.26	0.60	0.59	-0.47
	(0.14)	(0.18)	(0.18)	(0.10)	(0.12)	(0.11)
Canada	0.11	0.69	-0.01	0.41	0.77	0.12
	(0.15)	(0.12)	(0.15)	(0.13)	(0.08)	(0.16)
Denmark	0.18	0.02	-0.60	0.18	-0.26	-0.48
	(0.12)	(0.26)	(0.09)	(0.12)	(0.25)	(0.11)
Germany	0.04	0.86	-0.17	-0.01	0.71	0.01
	(0.13)	(0.06)	(0.14)	(0.15)	(0.09)	(0.16)
Italy	0.01	0.14	-0.33	-0.02	0.58	-0.24
	(0.10)	(0.15)	(0.14)	(0.11)	(0.09)	(0.14)
Japan	-0.49	-0.18	-0.37	-0.45	0.03	-0.60
	(0.11)	(0.25)	(0.18)	(0.11)	(0.22)	(0.10)
Norway	0.47	0.16	0.57	0.65	0.16	-0.63
	(0.11)	(0.16)	(0.10)	(0.08)	(0.19)	(0.08)
Sweden	-0.08	0.23	-0.38	0.15	0.30	-0.53
	(0.17)	(0.09)	(0.09)	(0.13)	(0.10)	(0.07)
U.K.	0.16	0.14	-0.72	0.26	0.20	-0.50
	(0.14)	(0.24)	(0.08)	(0.12)	(0.21)	(0.14)
U.S.	0.05	0.75	-0.25	0.22	0.72	-0.30
	(0.11)	(0.09)	(0.21)	(0.11)	(0.13)	(0.16)

Table 3: **The Correlation of Prices and Output.** **Numbers taken from Backus and Kehoe (1992). Numbers in parentheses are standard errors.*

	Point-in-time		Averaged	
N	$\lambda_{monthly}$	λ_{annual}	$\lambda_{monthly}$	λ_{annual}
4	-	36.04	-	11.03
8	-	27.9	-	7.05
12	30400	26.3	75200	6.57
24	38933	25.1	115520	6.33
52	53418	25.1	173804	7.27
365	44634	24.8	136396	6.18
730	43195	25.2	129586	6.32

Table 4: *The values of the smoothing parameter at the quarterly and at the annual frequency derived from the properties of termporally aggregated time-series*

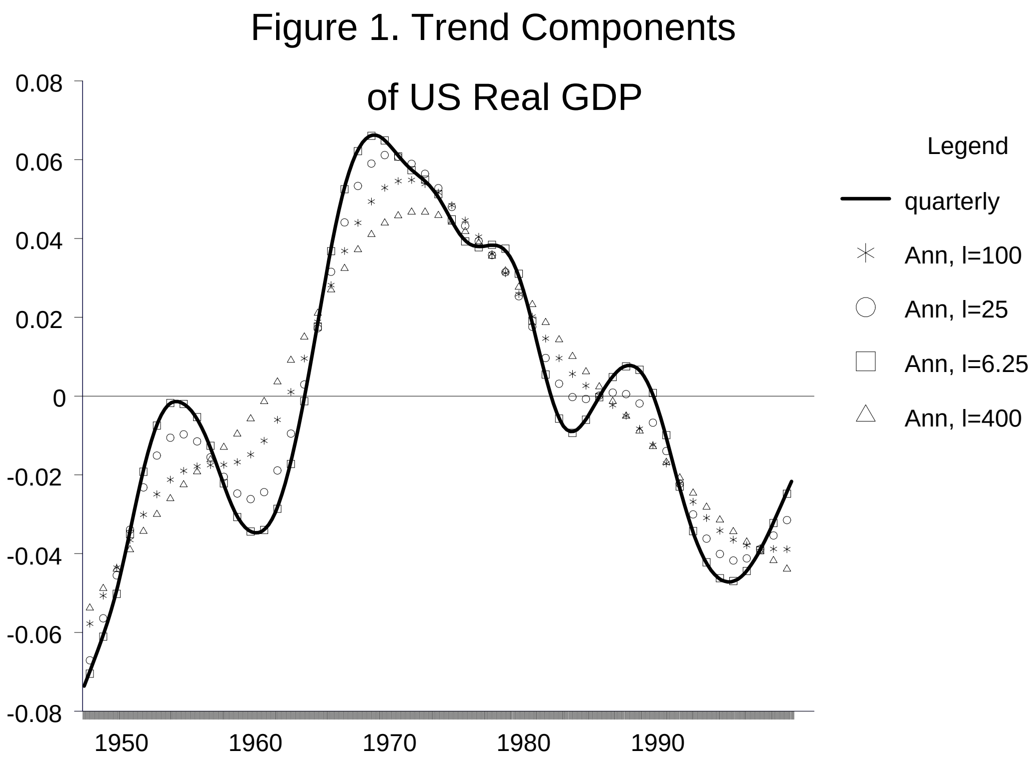

Figure 1: The Figure illustrates the HP-filter trend components of US real GDP sampled either at the quarterly frequency and using $\lambda_{quarterly} = 1600$ (the full drawn line) or at the annual frequency using alternative values for λ_{annual}. For $\lambda_{annual} = 6.25$ the trend components are practically identical. To make the figure clearer we have taken a linear trend out of the HP-filter trend components.