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Reversible Environmental Catastrophes with Disconnected Generations

Abstract

We study environmental policy in an economy-ecology model featuring multiple deterministic stable steady-state ecological equilibria. The economy-ecology does not settle in either of the deterministic steady states as the environmental system is hit by random shocks. Individual live for two periods and derive utility from the (stochastic) quality of the environment. They feature warm-glow preferences and therefore will engage in private abatement in order to slightly influence the stochastic process governing environmental quality. The government may also conduct abatement activities or introduce environmental taxes. We solve for the market equilibrium abstracting from public abatement and taxes and show that the ecological process may get stuck for extended periods of time fluctuating around the heavily polluted (low quality) deterministic steady state. These events are called environmental catastrophes. They are not irreversible, however, as the system typically switches back to the basin of attraction associated with the good (high quality) deterministic steady state. The paper also compares the stationary distributions for environmental quality and individuals' welfare arising under the unmanaged economy and in the first-best social optimum.

JEL-Codes: D600, E620, H230, H630, Q200, Q280, Q500.

Keywords: ecological thresholds, nonlinear dynamics, environmental policy, abatement, capital taxes.

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1 Introduction

"The window within which we may limit global temperature increases to 2 ^{o}C above preindustrial times is still open, but is closing rapidly. Urgent and strong action in the next two decades [...] is necessary if the risks of dangerous climate change are to be radically reduced."

Nicholas Stern, *Why Are We Waiting?* (2015, p. 32)

"...we are entering the Climate Casino. By this, I mean that economic growth is producing unintended but perilous changes in the climate and earth systems [which] will lead to unforeseeable and probably dangerous consequences. We are rolling the climatic dice, the outcome will produce surprises, and some of them are likely to be perilous. But we have just entered the Climate casino, and there is still time to turn around and walk back out."

William Nordhaus, *The Climate Casino* (2013, pp. 3-4)

"...I am a climate lukewarmer. That means I think recent global warming is real, mostly man-made and will continue but I no longer think it is likely to be dangerous and I think its slow and erratic progress so far is what we should expect in the future."

Matt Ridley, *The Times* newspaper (January 19, 2015)

Public commentators on climate change and, more generally, on current and future environmental issues seem to come in only two flavors. On the one hand, climate sceptics like bestselling popular science writer Matt Ridley and political scientist Bjørn Lomborg (and many others) tend to downplay the dangers and may even point at positive aspects of global warming. On the other hand, prominent environmental economists have assumed the mantle of whistle-blower and stress the immense risks current generations take with their own and future generations' environment and welfare. One of the reasons why no consensus has emerged up to this point is, of course, due to the fact that in normal times environmental changes are only gradual and slow (compared to an individual's life-span) and because the future is inherently stochastic and thus unknowable with certainty.

In this paper we present an explorative study in which we sketch what we consider to be important elements in the long-term evolution of the intertwined economic and ecological systems. In order to bring some structure to the debate we identify what we consider to be the four most crucial principles of model-based environmental policy analysis.

(P1) Generations are the relevant units of analysis. Sustainability is defined in the Brundtland Report (World Commission on Environment and Development, 1987, p. 43) as follows: "Sustainable development is development that meets the needs of the present without compromising the ability of future generations to meet their own needs." This suggests that the evaluation of environmental policy should be conducted in the context of an overlapping generations model with disconnected generations.

(P2) Abrupt environmental changes are possible. In recent years ecologists have discovered that nature does not always respond smoothly to gradual changes but instead may exhibit so-called "tipping points" in which dramatic environmental disasters occur (Scheffer *et al.*, 2001). Environmental economists have adopted the possibility of non-linearities in the response of the environmental system to economic developments. For a recent symposium on the economics of tipping points, see de Zeeuw and Li (2016).

(P3) Both the economy and the ecological system are inherently stochastic. Indeed, as is stressed by both Stern and Nordhaus in the quotes given above, global warming should not be seen as a deterministic process but rather should be recognized as being inherently stochastic in nature. A suitable model of environmental policy must thus explicitly recognize the fact that both private and public decision making takes place in a world hit by random shocks.

(P4) Individuals care for the environment but not very strongly. On the one hand, environmental quality has strong public good features so that rational individuals tend to free ride on it. On the other hand, we believe that (at least some) people do get a "warm glow" from cleaning up their local parks and beaches, even if it is merely to be seen "doing the right thing" by their neighbours and friends. A modest amount of private abatement does take place in reality and we capture this phenomenon by adopting the insights of Andreoni (1988, 1989, 1990) and Andreoni and Levinson (1990).

The objective of this paper is to study environmental policy using a highly stylized conceptual model which can accommodate all principles (P1)–(P4) simultaneously. In order to capture Principle (P1) we employing an explicit general equilibrium overlapping-generations framework of the economy-ecology interaction. By adopting a closed-economy perspective we capture the notion of global interactions between the economy and the environment. We also assume that the generations of cohorts populating the planet are disconnected with each other, i.e. we abstract from voluntary intergenerational transfers from parent to child (and vice versa). The disconnectedness of generations ensures that current generations will not voluntarily provide monetary transfers to future generations to compensate the latter for the environmental sins committed by the former.

Principle (P2) is accommodated by postulating a nonlinear environmental regeneration function which includes tipping points and multiple stable (deterministic) equilibria. In order to avoid modeling environmental policy as a "one-shot game" (in which only one irreversible catastrophe can occur), we assume that the resulting hysteresis in environmental quality is reversible, albeit at potentially very high cost. In technical terms we recast our earlier deterministic and continuous-time studies to a discrete-time stochastic setting. See Heijdra and Heijnen (2013, 2014).

Principle (P3) is captured, though partially, by including stochastic shocks to the state equation for environmental quality. Although random shocks to the economic system are also potentially important to the proper conduct of environmental policy, we abstract from such shocks in the present paper to keep the analysis manageable. By assuming that ecological disasters are potentially reversible (via (P2)), we find that in a stochastic setting multiple low-environmental-quality *epochs* of varying duration can materialize, something which is impossible in the somewhat restrictive stochastic single-disaster framework of Tsur and Zemel (2006), Polasky et al. (2011), and many others.

Finally, Principle (P4) is included by introducing a "warm-glow" mechanism into the utility function of individual agents. This ensures that utility maximizing individuals engage in a modest amount of private abatement (because it makes them feel good) but otherwise free ride of the abatement activities by other individuals and (potentially) the government. So in our model environmental quality is a non-excludable and non-rival public good but there is some private provision going on at all times. By construction we assume that the warm-glow motive is relatively weak so that there is typically "too little" environmental abatement in the absence of an active public abatement stance by the government.

The paper is structured as follows. In Section 2 we present a deterministic version of our model (and thus exclude Principle (P3) in doing so). Individual live for two periods, youth and old-age, consume in both period, work only in the first period, and enjoy environmental quality in the second period. Explicit saving during youth takes the form of capital accumulation whilst implicit saving occurs in the form of private abatement which augments the future environmental quality. Firms use capital and labour to produce a homogeneous commodity which can be used for consumption, private and public abatement, and investment.

In Section 3 we assume that the policy maker does neither engage in public abatement nor employs Pigouvian pollution taxes. We label this case the Deterministic Unmanaged Market Economy (DUME). We show that the model can be condensed into a stable two-equation system of difference equations in the capital intensity and environmental quality. Since both private saving and private abatement depend on both state variables, the dynamic system is fully simultaneous so that analytical results are hard to come by. In order to visualize and quantify the key properties of the model we develop a plausible calibration. The numerical model implies that the effect of environmental quality on the macro-economic equilibrium is quite weak unless the ecological system is stuck in a highly polluted state. In "normal times" individuals simply do not care enough about the environment for them to be influenced by even sizeable fluctuations in environmental quality. In this section we consider two prototypical environmental regeneration function. When the feedback between current and future environmental quality is linear then the system will ultimately settle in a unique steady state for the capital intensity and environmental quality. In contrast, when the feedback is described by a nonlinear regeneration function of the right type, then there exist two welfare-rankable steady-state equilibria. Whilst the capital intensity differs little between the two equilibria, in the low-welfare equilibrium environmental quality is rather low whilst it is rather high in the high-welfare equilibrium. To prepare for things to come, Section 3 concludes by computing the deterministic (first-best) social optimum (DSO) that is chosen by a dynamically consistent social planner. Not surprisingly, starting from either of the possible equilibria in the DUME state with a nonlinear regeneration function, such a planner will select a transition path that will result in a unique steady state featuring a high level of environmental quality.

Section 4 constitutes the core of our paper. In this section we re-instate Principle (P3) and study the economy-environment interaction in an inherently stochastic setting. In particular we assume that the state equation for environmental regeneration is hit by random shocks and that the regeneration function is nonlinear and features tipping points. During youth, individuals face uncertainty about the environmental quality they will enjoy during old-age and they take this into account when making optimal decisions concerning saving, consumption, and private abatement. There is some precautionary saving and private abatement given that the utility function features prudence. If the government does not conduct any environmental policy at all then the system will settle in a stochastic steady state which we label the Stochastic Unmanaged Market Economy (SUME). Very long-run simulations of the SUME model show that the system displays clear and often long-lasting epochs during which it fluctuates in the vicinity of either the low-welfare or high-welfare deterministic steady state. This is a clear demonstration of the reversible hysteresis that is a feature of the nonlinear model. Whilst the fluctuations in the economic variables are quite small (both within and between epochs), the variability of environmental quality is quite substantial. Private abatement activities are larger during a low-welfare epoch but they are not high enough to force the system back to the high-welfare basin of attraction.

In the second part of Section 4 we compute the stochastic (first-best) social optimum (SSO) that is chosen by a dynamically consistent social planner operating under the same degree of uncertainty as the public about future environmental quality. Such a planner computes state-dependent policy functions for private and public abatement, consumption by young and old, the future capital intensity, and the deterministic part of future environmental quality. Evaluated for the average capital intensity, the policy function for public abatement is strongly decreasing in pre-existing environmental quality whilst the one for private abatement displays the opposite pattern. This seemingly paradoxical result is explained by our maintained assumption that public abatement is more efficient than private abatement. Since the social planner operates in a stochastic environment the SSO constitutes a stochastic process for all key variables. To characterize the key features of this stochastic process we compute probability density functions for public and private abatement, the capital intensity, and environmental quality. Just as in the deterministic case the social planner eliminates the low-quality equilibrium by its policies, i.e. the PDF of environmental quality is centered tightly around the high-quality state of the environment. The comparison of the PDFs for environmental quality under the SUME and SSO reveals that the former is bimodal and the latter is unimodal. The PDF for expected lifetime utility at birth shows a similar pattern. In addition, in terms of lifetime utility there is a huge degree of inequality between lucky and unlucky generations. Behind the veil of ignorance individuals are vastly better off in a world fine-tuned by a social planner than under the unmanaged economy.

In the final part of Section 4 we investigate whether and to what extent a simple linear feedback policy rule for public abatement can improve welfare for current and future generations. The particular policy rule we consider stipulates that public abatement is a downward sloping linear function of the pre-existing environmental quality. To parameterize this function we fit a straight line though the relevant part of the SSO policy function evaluated at the average capital intensity. This rule obviously falls short of the first-best scenario, both because it is linear and because it does not address any issues other than public abatement by its very design. Surprisingly, however, the linear feedback policy rule does quite well. The PDFs for both environmental quality and utility at birth are shifted to the left somewhat but they remain unimodal. This suggests that a simple constitutional rule for public abatement (binding the hands of future opportunistic politicians) may have some attractive features.

Finally, in Section 5 we offer a brief summary of the main results and offer some thoughts on future work. Three appendices present some technical details.

1.1 Relationship with the existing literature

Our paper contributes to an ongoing literature on the interactions between the aggregate economy and the environment. One of the earliest contributions to that literature is the paper by John and Pecchenino (1994). They employ a deterministic two-period overlapping-generations (OG) model and assume that the environmental state equation features a linear regeneration function thus precluding tipping points. In terms of the principles mentioned above, only (P1) is addressed. Environmental quality is modeled as a pure public good but they abstract from the free-rider problem within a generation by assuming that a benevolent government sets taxes on the young and provides the right amount of environmental quality when these agents are old in the next period and derive utility from it.

Prieur (2009) generalizes the John-Pecchenino model by assuming that the environmental regeneration function is hump-shaped and becomes zero beyond a certain critical level of the pollution stock. As a result his model features a tipping point and gives rise to multiple equilibria. Hence both principles (P1) and (P2) are addressed. The young agent engages in private abatement and takes into account only what his/her green investment does to environmental quality when old. The public good nature of abatement is thus again ignored.

The nonlinear ecological dynamics described by Scheffer *et al.* (2001) is often referred to as Shallow-Lake Dynamics (SLD hereafter). For overviews of the SLD approach, see Muradian (2001), Mäler *et al.* (2003), and Brock and Starrett (2003). For economic applications of SLD, see Heijdra and Heijnen (2013, 2014) and the references therein.

In a number of papers Tsur and Zemel (1996, 1998, 2006) introduce a specific type of uncertainty into the environmental model, namely event uncertainty. In their approach there is a non-zero probability of an environmental disaster occurring at any time. Since this probability depends positively on the pollution stock, the social planner will take this mechanism into account when formulating an optimal environmental policy. The Tsur-Zemel papers have triggered a large and ongoing literature with prominent contributions by Polasky et al. (2011), Lemoine and Traeger (2014), van der Ploeg (2014), and van der Ploeg and de Zeeuw (2016, forthcoming). In this literature principles (P2) and (P3) are dealt with but (P1) is ignored. Heijnen and Dam (forthcoming) adds (P4) but treats (P2) and (P3) in a parsimonious manner only.

Although it is completely different in focus, the paper that comes closest to ours is Grass et al. (2015). They study a stochastic optimal control problem of the shallow-lake type in which the state equation for the pollution stock is continuously hit by random shocks. A social planner controls the usage of fertilizers and balances the conflicting interests of farmers (who indirectly benefit from pollution) and tourists (who are harmed by pollution). Depending on the noise intensity the optimal policy gives rise to a unimodal or bimodal probability density function for environmental quality. Whereas our model always yields a unimodal distribution of environmental quality, the analysis of Grass et al. (2015) suggests that this conclusions is dependent on both the functional form of the abatement technology and the parameterization of the model.

2 A deterministic model

In this section we develop and analyze a deterministic version of our overlapping-generations model featuring two-period lived individuals who voluntarily engage in moderate amounts of private abatement, in part because such activities gives them a 'warm glow'. This approach was pioneered by Andreoni (1989, 1990) and is applied to the environment here. Environmental quality is negatively affected by the output produced in the economy, but both private and public abatement can be used to clean up the environmental mess created by human activities.

2.1 Consumers

Each period a large cohort of size L of identical individuals is born.[1] Each agent lives for two periods, works full-time during the first period of life (termed "youth") and is retired in the second period ("old age"). Lifetime utility of individual i born at time t is given by:

$$\Lambda_t^i \equiv U(c_t^{y,i}) + \chi V(m_t^i) + \beta \left[U(c_{t+1}^{o,i}) + \zeta W(Q_{t+1}) \right], \tag{1}$$

where $c_t^{y,i}$ and $c_{t+1}^{o,i}$ are, respectively, consumption during youth and old age, m_t^i represents private environmental abatement activities (χ is the 'warm-glow' parameter such that $\chi > 0$), Q_{t+1} is the quality of the environment during old age (a non-excludable and non-rival public good, with $\zeta > 0$), and $\beta \equiv 1/(1+\rho)$ is the discount factor where $\rho > 0$ is the pure rate of time preference. The felicity functions exhibit the usual properties, i.e. $U'(x) > 0$, $\lim_{x \to 0} U'(x) =$

[1] We follow John and Pecchenino (1994) by assuming that the population is constant (and equal to $2L$).

$+\infty$, $U''(x) < 0$, $V'(x) > 0$, $\lim_{x\to 0} V'(x) = +\infty$, $V''(x) < 0$, $W'(x) > 0$, $\lim_{x\to 0} W'(x) = +\infty$, and $W''(x) < 0$. Individuals have no bequest motive and, therefore, attach no utility to savings that remain after they die. Note that, in contrast to John and Pecchenino (1994) and Prieur (2009), we assume that the agent voluntarily engages in activities which are aimed at improving environmental quality and recognizes his/her own (small) effect on total abatement.[2]

The agent's budget identities for youth and old age are given by:

$$c_t^{y,i} + s_t^i + m_t^i = w_t - \tau_t, \tag{2}$$

$$c_{t+1}^{o,i} = (1 + r_{t+1}) s_t^i, \tag{3}$$

where w_t is the wage rate, r_t is the interest rate, s_t^i denotes the level of savings, and τ_t is the lump-sum tax charged by the government during youth. For reasons of analytical and computational convenience we abstract from taxation of old-age individuals. Agents are blessed with perfect foresight regarding all future variables. The transition equation for environmental quality takes the following form:

$$Q_{t+1} = H(Q_t) - D_t, \tag{4}$$

where $H(Q_t)$ is an increasing function capturing the regenerative capacity of the environment ($H'(Q_t) > 0$), and D_t is the pollution flow resulting from economic activities. Throughout the paper we assume that the pollution flow is proportional to aggregate output produced in the economy (denoted by Y_t):

$$D_t = \xi Y_t e^{-\gamma M_t - \eta G_t}, \qquad \xi > 0. \tag{5}$$

In equation (5), G_t is public abatement, $M_t \equiv \sum_{i=1}^{L} m_t^i$ is total private abatement, and γ and η are constant positive parameters. By entering these abatement activities exponentially we incorporate the notion of convex adjustment costs, i.e. $\partial D_t/\partial G_t = -\eta D_t < 0$, $\partial^2 D_t/\partial G_t^2 = \eta^2 D_t > 0$, $\partial D_t/\partial M_t = -\gamma D_t < 0$, $\partial^2 D_t/\partial M_t^2 = \gamma^2 D_t > 0$. We assume that the government is more efficient at abatement than private individuals are, i.e. $\eta > \gamma > 0$. Since output is strictly positive the flow of dirt is guaranteed to be positive also, i.e. $D_t > 0$. Two prototypical specifications for the regeneration function, $H(Q_t)$, are formulated and discussed below.

Agent i chooses $c_t^{y,i}$, $c_{t+1}^{o,i}$, s_t^i, and m_t^i in order to maximize expected lifetime utility (1) subject to the budget identities (2)–(3) and the environmental transition function (4). The individual takes as given factor prices, taxes, aggregate output, as well as the abatement expenditures by other individuals, $M_t^{\neg i} \equiv \sum_{j\neq i}^{L} m_t^j$, and the government, G_t. We define Z_t as:

$$Z_t \equiv \xi Y_t e^{-\gamma M_t^{\neg i} - \eta G_t}, \tag{6}$$

[2]Both studies abstract from the free-rider problem within a generation. John and Pecchenino (1994, p. 1396) provide an interpretation for this assumption and relate it to Lindahl pricing.

note that $D_t = Z_t e^{-\gamma m_t^i}$, and find that the key first-order conditions are:

$$U'(c_t^{y,i}) = \beta(1 + r_{t+1})U'(c_{t+1}^{o,i}), \tag{7}$$
$$U'(c_t^{y,i}) = \chi V'(m_t^i) + \beta\gamma\zeta Z_t e^{-\gamma m_t^i} W'(H(Q_t) - Z_t e^{-\gamma m_t^i}). \tag{8}$$

The optimal savings decision is implicitly characterized by the consumption Euler equation in (7). It ensures that the marginal rate of substitution between future and current consumption is equated to the intertemporal price of future consumption. The optimal abatement choice is characterized by (8). Here the trade-off is between giving up some current consumption (left-hand side) in order to experience a warm glow (first term on the right-hand side) and to obtain a slight gain in future environmental utility (second term).

In the remainder of this paper we assume that the three felicity functions featuring in lifetime utility are logarithmic, i.e. $U(x) = V(x) = W(x) = \ln x$. Since all agents in a given cohort are identical, it follows that they make the same choices, i.e. $c_t^{y,i} = c_t^y$, $s_t^i = s_t$, $m_t^i = m_t$, and $c_{t+1}^{o,i} = c_{t+1}^o$ for all i. The optimal choices for c_t^y, m_t, and c_{t+1}^o are characterized by:

$$\frac{c_{t+1}^o}{c_t^y} = \beta(1 + r_{t+1}), \tag{9}$$
$$\frac{\chi}{m_t} + \frac{\beta\gamma\zeta e^{-\gamma L m_t} D_t^g}{H(Q_t) - e^{-\gamma L m_t} D_t^g} = \frac{1}{c_t^y}, \tag{10}$$
$$c_t^y + m_t + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t - \tau_t, \tag{11}$$
$$D_t^g = \xi L y_t e^{-\eta L g_t}, \tag{12}$$

where $y_t \equiv Y_t/L$ and $g_t \equiv G_t/L$ are, respectively, output and public abatement per worker, and D_t^g is the dirt flow that would result in the absence of private abatement (the so-called *gross* dirt flow). Ceteris paribus $w_t - \tau_t$, r_{t+1}, Q_t, and D_t^g, the optimal choices made by the individual can be explained with the aid of Figure 1. In the top panel the curve labeled PA_0 represents equation (10) and states the optimal level of private abatement for different levels of youth consumption. The curve labeled HBC_0 is the household budget constraint. It is obtained by substituting (9) into (11):

$$m_t + (1 + \beta)c_t^y = w_t - \tau_t. \tag{13}$$

Because (a) the logarithmic felicity functions imply a unitary intertemporal substitution elasticity and (b) agents do not receive any wage income or pay taxes during old-age, the budget constraint is independent of the future real interest rate. The optimum choices for m_t and c_t^y are located at point E_0 in the top panel, and can be written as $m_t = \mathbf{m}(w_t - \tau_t, Q_t, D_t^g)$ and $c_t^y = \mathbf{c}^y(w_t - \tau_t, Q_t, D_t^g)$. The implied savings function is written as $s_t = \mathbf{s}(w_t - \tau_t, Q_t, D_t^g)$. In the bottom panel of Figure 1 EE_0 depicts the consumption Euler equation (9). For future reference we write $c_{t+1}^o = \beta(1 + r_{t+1})\mathbf{c}^y(w_t - \tau_t, Q_t, D_t^g)$.

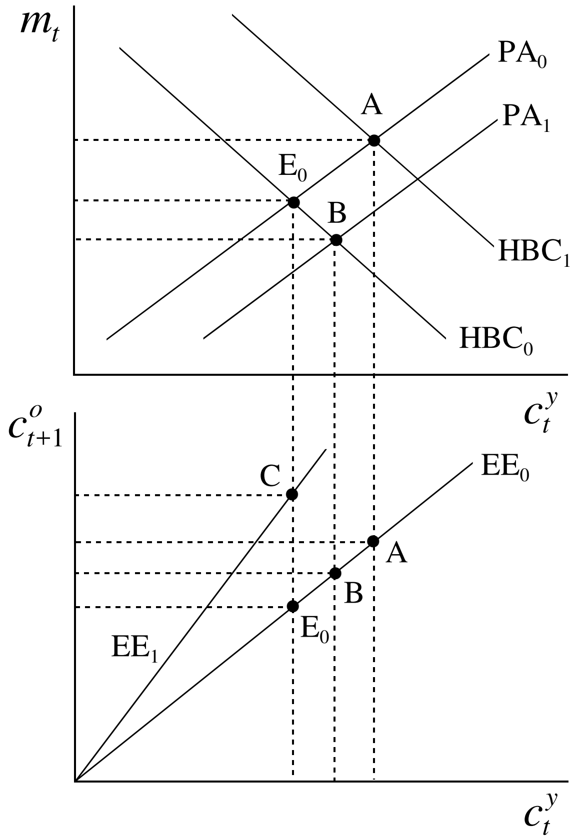


Figure 1: Privately optimal consumption and private abatament

The comparative static effects of the various determinants of m_t, c^y_t, and s_t can be illustrated with the aid of Figure 1. First, an increase in w_t (or decrease in τ_t) shifts the budget equation from HBC$_0$ to HBC$_1$ so that the new private optimum occurs at point A. It follows that m_t, c^y_t, and c^o_{t+1} are normal goods, i.e. $0 < \mathbf{m}_w \equiv \partial\mathbf{m}(w_t - \tau_t, Q_t, D^g_t)/\partial(w_t - \tau_t) < 1$, $0 < \partial\mathbf{c}^y(w_t - \tau_t, Q_t, D^g_t)/\partial(w_t - \tau_t) < 1$, and $\partial c^o_{t+1}/\partial(w_t - \tau_t) > 0$. Saving also increases, i.e. $0 < \mathbf{s}_w \equiv \partial\mathbf{s}(w_t - \tau_t, Q_t, D^g_t)/\partial(w_t - \tau_t) < 1$. Second, an increase in the future interest rate has no effect on the optimal choices for m_t, c^y_t, and s_t but it leads to an increase in c^o_{t+1}. In the bottom panel of Figure 1 the Euler equation rotates from EE$_0$ to EE$_1$ and the private optimum shifts from point E$_0$ to C. Third, an increase in Q_t and a decrease in D^g_t both lead to a downward shift in the private abatement curve, say from PA$_0$ to PA$_1$ in the top panel of Figure 1. The optimum shifts from E$_0$ to B in both panels, and it follows that $\mathbf{m}_Q \equiv \partial\mathbf{m}(w_t - \tau_t, Q_t, D^g_t)/\partial Q_t < 0$, $\mathbf{m}_D \equiv \partial\mathbf{m}(w_t - \tau_t, Q_t, D^g_t)/\partial D^g_t > 0$, $\partial\mathbf{c}^y(w_t - \tau_t, Q_t, D^g_t)/\partial Q_t > 0$, $\partial\mathbf{c}^y(w_t - \tau_t, Q_t, D^g_t)/\partial D^g_t < 0$, $\mathbf{s}_Q \equiv \partial\mathbf{s}(w_t - \tau_t, Q_t, D^g_t)/\partial Q_t > 0$, $\mathbf{s}_D \equiv \partial\mathbf{s}(w_t - \tau_t, Q_t, D^g_t)/\partial D^g_t < 0$, $\partial c^o_{t+1}/\partial Q_t > 0$ and $\partial c^o_{t+1}/\partial D^g_t < 0$.

2.2 Firms

The firm sector is perfectly competitive and operates under constant returns to scale. The representative firm hires capital K_t and labour N_t in order to produce homogeneous output Y_t. For simplicity the technology available to the firm is of the Cobb-Douglas form:

$$Y_t = \Omega K_t^{\alpha} N_t^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{14}$$

where α is the efficiency parameter of capital and Ω is the aggregate level of technology in the economy. Factor demands of the firm are given by the following marginal productivity conditions:

$$w_t = (1-\alpha)\,\Omega k_t^{\alpha}, \tag{15}$$
$$r_t + \delta = \alpha \Omega k_t^{\alpha-1}, \tag{16}$$

where $k_t \equiv K_t/L$ is the capital intensity, $\delta > 0$ is the depreciation rate, and we have incorporated labour market equilibrium, $N_t = L$. Output per worker is thus given by:

$$y_t = f(k_t) \equiv \Omega k_t^{\alpha}, \tag{17}$$

where $y_t \equiv Y_t/L$.

2.3 Other model features

The economy-wide resource constraint per worker can be written as:

$$y_t + (1-\delta)\,k_t = c_t^y + c_t^o + m_t + g_t + k_{t+1}, \tag{18}$$

where $g_t \equiv G_t/L$ is public abatement spending per worker. Total available resources, consisting of output and the undepreciated part of the capital stock, are spent on consumption (by young and old individuals), on abatement (by young agents and the government), and on the future stock of capital. Total saving by the young determines the future capital stock, i.e. $Ls_t = K_{t+1}$ or:

$$k_{t+1} = s_t. \tag{19}$$

In the absence of public debt, the government budget constraint per worker can be written as:

$$g_t = \tau_t^y. \tag{20}$$

The policy maker uses public abatement as its environmental instrument and balances the budget by choice of the lump-sum tax on the young.

For future reference the deterministic overlapping-generations model developed in this section has been summarized in Table 1. Equation (T1.1) is obtained by substituting the savings function (with (20) imposed), $\mathbf{s}(w_t - g_t, Q_t, D_t^g)$, into the capital accumulation equation (19). Equation (T1.2) is obtained by using (4)–(5) and (12), and substituting the private abatement function, $\mathbf{m}(w_t - g_t, Q_t, D_t^g)$. Equations (T1.3)–(T1.5) restate, respectively, (17), (15), and (12).

Table 1: The deterministic environmental overlapping-generations model

$$k_{t+1} = \mathbf{s}(w_t - g_t, Q_t, D_t^g), \tag{T1.1}$$

$$Q_{t+1} = H(Q_t) - e^{-\gamma L \mathbf{m}(w_t - g_t, Q_t, D_t^g)} D_t^g \tag{T1.2}$$

$$y_t = \Omega k_t^{\alpha} \tag{T1.3}$$

$$w_t = (1-\alpha)\, y_t \tag{T1.4}$$

$$D_t^g = \xi L y_t e^{-\eta L g_t} \tag{T1.5}$$

Variables:		*Parameters:*	
k_t	capital intensity	Ω	productivity parameter ($\Omega > 0$)
Q_t	environmental quality	α	efficiency parameter of capital ($0 < \alpha < 1$)
y_t	output per worker	γ	private abatement parameter ($\gamma > 0$)
g_t	public abatement per worker	η	public abatement parameter ($\eta > \gamma$)
w_t	wage rate	ξ	output dirt parameter ($\xi > 0$)
D_t^g	gross dirt flow	L	number of workers

3 Economic-environmental dynamics in a deterministic world

3.1 Unmanaged market equilibrium

Despite its highly stylized nature the model stated in Table 1 incorporates a rich array of interactions between the environment and the economic process. Indeed, the fundamental system of difference equations for the capital intensity and environmental quality is fully characterized by:

$$k_{t+1} = \mathbf{s}((1-\alpha)\,\Omega k_t^{\alpha} - g_t, Q_t, D_t^g), \tag{21}$$

$$Q_{t+1} = H(Q_t) - e^{-\gamma L \mathbf{m}((1-\alpha)\Omega k_t^{\alpha} - g_t, Q_t, D_t^g)} D_t^g, \tag{22}$$

$$D_t^g = \xi L \Omega k_t^{\alpha} e^{-\eta L g_t}. \tag{23}$$

Because young individuals care for the environmental quality they will enjoy during old-age ($\mathbf{s}_Q > 0$), the dynamics of the capital intensity is affected by the current state of the environment, Q_t. Furthermore, the dynamics of environmental quality is affected by the current capital intensity, k_t, both because of its effect on current output and wages, and because young agents increase the level of private abatement if the gross pollution flow increases ($\mathbf{m}_D > 0$).

We assume that public abatement is equal to zero and that the system features a steady state equilibrium denoted by (k^*, Q^*). Local dynamic around the steady state can then–at least in principle–be studied with the aid of the linearized system:

$$\begin{bmatrix} k_{t+1} - k^* \\ Q_{t+1} - Q^* \end{bmatrix} = \Delta \begin{bmatrix} k_t - k^* \\ Q_t - Q^* \end{bmatrix}, \tag{24}$$

where the Jacobian matrix is defined as:

$$\Delta \equiv \begin{bmatrix} [(1-\alpha)\mathbf{s}_w + \xi L \mathbf{s}_D]\,(r^* + \delta) & \mathbf{s}_Q \\ [\gamma(1-\alpha)\mathbf{m}_w + \gamma\xi L \mathbf{m}_D - 1/(Lf(k^*))]\, LD^*(r^* + \delta) & H'(Q^*) + \gamma LD^* \mathbf{m}_Q \end{bmatrix}, \tag{25}$$

and where $\mathbf{s}_w$, $\mathbf{s}_Q$, $\mathbf{s}_D$, $\mathbf{m}_w$, $\mathbf{m}_Q$, and $\mathbf{m}_D$ denote the partial derivatives of the savings and abatement functions with respect to the argument in the subscript. We recall from the preceding discussion that $0 < \mathbf{s}_w, \mathbf{m}_w < 1$, $\mathbf{s}_Q > 0$, $\mathbf{s}_D < 0$, $\mathbf{m}_Q < 0$,and $\mathbf{m}_D > 0$. Since both k_t and Q_t are predetermined variables, stability requires the characteristic roots of Δ to lie inside the unit circle.

As is clear from the structure of the Jacobian matrix in (25) the model is too complicated for it to yield clear-cut analytical results. For that reason we adopt a plausible parameterization of the model and use it to numerically study the interaction between the environment and the economy in the remainder of this paper. Although it is not difficult to come up with plausible values for the purely economic parameters (such as α, β and δ) it is much harder to assign numbers to the structural parameters characterizing the environmental effects in the model (γ, ζ, η, ξ, and χ). We document our parameterization approach in detail in Appendix A. Essentially we formulate targets relating to economic and environmental variables that must be met in the unmanaged market economy.

Table 2 provides an overview of the structural parameters of the model. Each period is assumed to last for 30 years and there are one hundred individuals in the economy ($L = 100$). The discount factor β is based on an annual rate of pure time preference (ρ_a) of four percent. The efficiency parameter of capital in the production function is equal to $\alpha = 0.3$. The constant in the production function is set such that the target level of output equals unity. Furthermore, the capital depreciation rate is chosen such that the target (annual) interest rate of 2.5 percent is attained.

In order to prepare for things to come, we first visualize some aspects of the parameterized steady-state market equilibrium of the unmanaged economy *conditional* on the steady-state level of environmental quality $\hat{Q}$ (and without public abatement, $g_t = 0$). The advantage of doing so is that it allows us to derive insights into the basic mechanisms at work in the unmanaged economy without having to postulate a specific functional form for the regeneration function $H(Q_t)$. The system characterizing the conditional unmanaged market equilibrium is given by:

$$\hat{c}^o = \beta(1 + \hat{r})\hat{c}^y, \tag{26}$$

$$\frac{1}{\hat{c}^y} = \frac{\chi}{\hat{m}} + \frac{\beta\gamma\zeta\hat{D}}{\hat{Q}}, \tag{27}$$

$$\hat{w} = \hat{c}^y + \hat{k} + \hat{m}, \tag{28}$$

$$\hat{c}^o = (1+\hat{r})\hat{k}, \tag{29}$$

$$\hat{r} = \alpha\Omega\hat{k}^{\alpha-1} - \delta, \tag{30}$$

$$\hat{w} = (1-\alpha)\Omega\hat{k}^{\alpha}, \tag{31}$$

$$\hat{D} = \xi L\Omega\hat{k}^{\alpha}e^{-\gamma L\hat{m}}, \tag{32}$$

where hats denote steady-state values and where the endogenous variables are $\hat{c}^o$, $\hat{c}^y$, $\hat{m}$, $\hat{k}$, $\hat{w}$, $\hat{r}$, and $\hat{D}$.

We assume that environmental quality lies in the interval $[0, \bar{Q}]$ with $Q_t \approx 0$ representing a situation comparable to Dante's *Inferno* whilst $Q_t = \bar{Q}$ can be seen as characterizing the pristine environment. In Figure 2 we depict the conditional steady state for a number of key variables (noting that output per worker is given by $\hat{y} = \Omega\hat{k}^{\alpha}$). The main lesson to be learned from the figure is unambiguous. For all but extremely low values of steady-state environmental quality, $\hat{k}$, $\hat{y}$, $\hat{m}$, $\hat{c}^y$, and $\hat{c}^o$ are virtually independent of the value of $\hat{Q}$. Utility-maximizing individuals will only engage in a large amount of private abatement (and cut back their saving a lot to do so) if push comes to shove, i.e. if the environmental quality comes close to diabolical levels. For any other values of $\hat{Q}$, such individuals will conduct a modest amount of private abatement in order to satisfy their warm-glow motive for doing so.

Whilst the Figure 2 is useful to illustrate some mechanisms at work, it does not pin down which equilibrium will actually be attained. In order to determine the equilibrium in the unmanaged economy we must adopt a specific functional form for the environmental regeneration function $H(Q_t)$. In the next two subsections we will consider two prototypical regeneration functions, a linear one (giving rise to a unique steady-state equilibrium) and a nonlinear one (yielding multiple steady-state equilibria).

3.1.1 Linear environmental dynamics

In this subsection we assume that the environmental regeneration function is linear:

$$H(Q_t) \equiv \theta\bar{Q} + (1-\theta)Q_t, \tag{33}$$

where $\bar{Q} > 0$ is the maximum level of environmental quality (pristine nature), and θ is the adjustment parameter satisfying $0 < \theta < 1$. In our numerical simulations we assume that the annual rate of environmental regeneration (θ_a) is two percent (see Table 2), implying a relatively slow rate of adjustment in environmental quality (compared to the speed of adjustment in the economic process). By using (33) in (4) and imposing the steady state we find:

$$\hat{Q} = \bar{Q} - \frac{1}{\theta}\hat{D}. \tag{34}$$

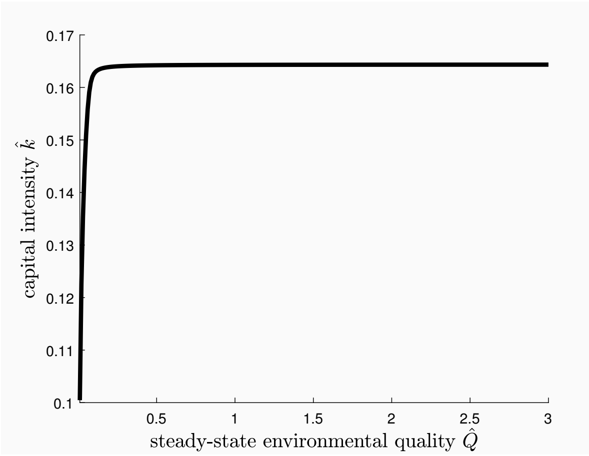
Table 2: Structural parameters

Economic parameters				
β	discount factor		0.3083	
L	young cohort size		100.0000	
ρ_a	annual time preference (percent)		4.0000	
α	capital share parameter		0.3000	
Ω	production function constant	c	1.7190	
δ_a	annual capital depreciation rate (percent)	c	4.2468	
δ	capital depreciation factor	c	0.7280	
Environmental parameters				
χ	taste parameter for private abatement	c	4.8584	10^{-3}
ζ	taste parameter for future environmental quality		25.0000	
γ	environmental dirt-private-abatement parameter	c	7.5807	10^{-2}
η	environmental dirt-public-abatement parameter	c	8.4230	10^{-2}
ξ	environmental dirt-output parameter	c	2.3190	10^{-3}
θ_a	annual rate of environmental regeneration (percent)		2.0000	
θ	environmental regeneration factor		0.4545	
$\bar{Q}$	maximum environmental quality		3.0000	

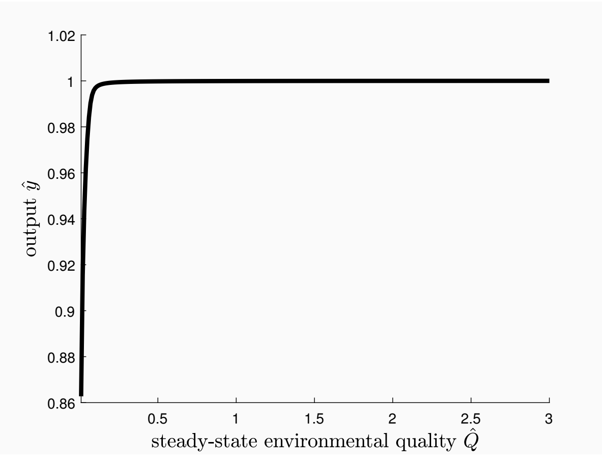
Note See Appendix A for details on the parameterization approach. The parameters labeled 'c' are calibrated as is explained in the appendix. The remaining parameters are postulated a priori. The values for δ, θ, and $\beta \equiv 1/(1+\rho)$ follow from, respectively, δ_a, θ_a, and ρ_a, by noting that each model period represents 30 years.

Figure 2: The steady-state unmanaged economy conditional on environmental quality

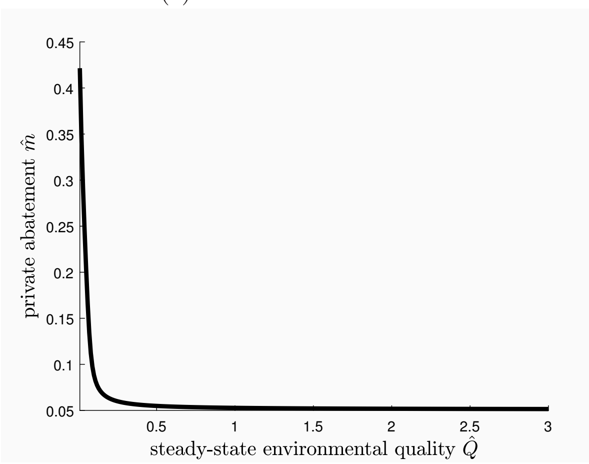
(a) Capital intensity $\hat{k}$



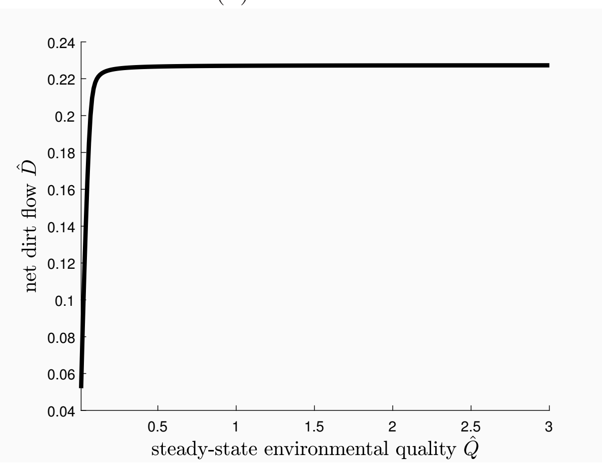
(b) Output $\hat{y}$



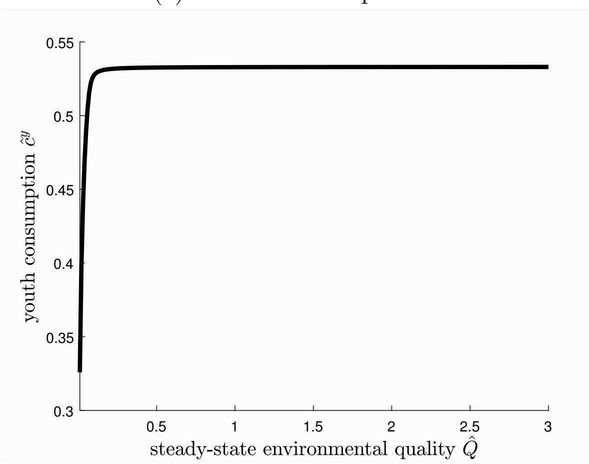
(c) Private abatement $\hat{m}$



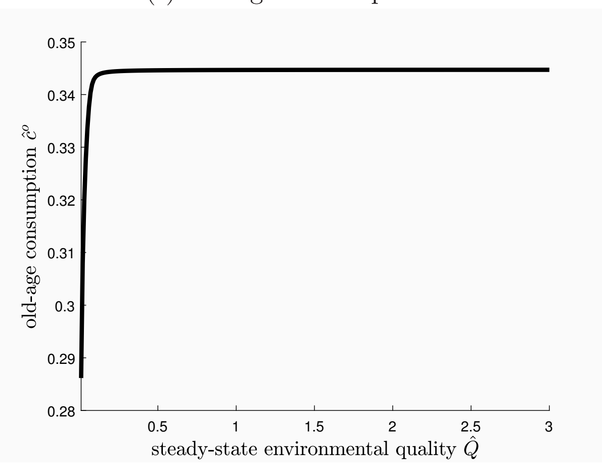
(d) Dirt flow $\hat{D}$



(e) Youth consumption $\hat{c}^y$

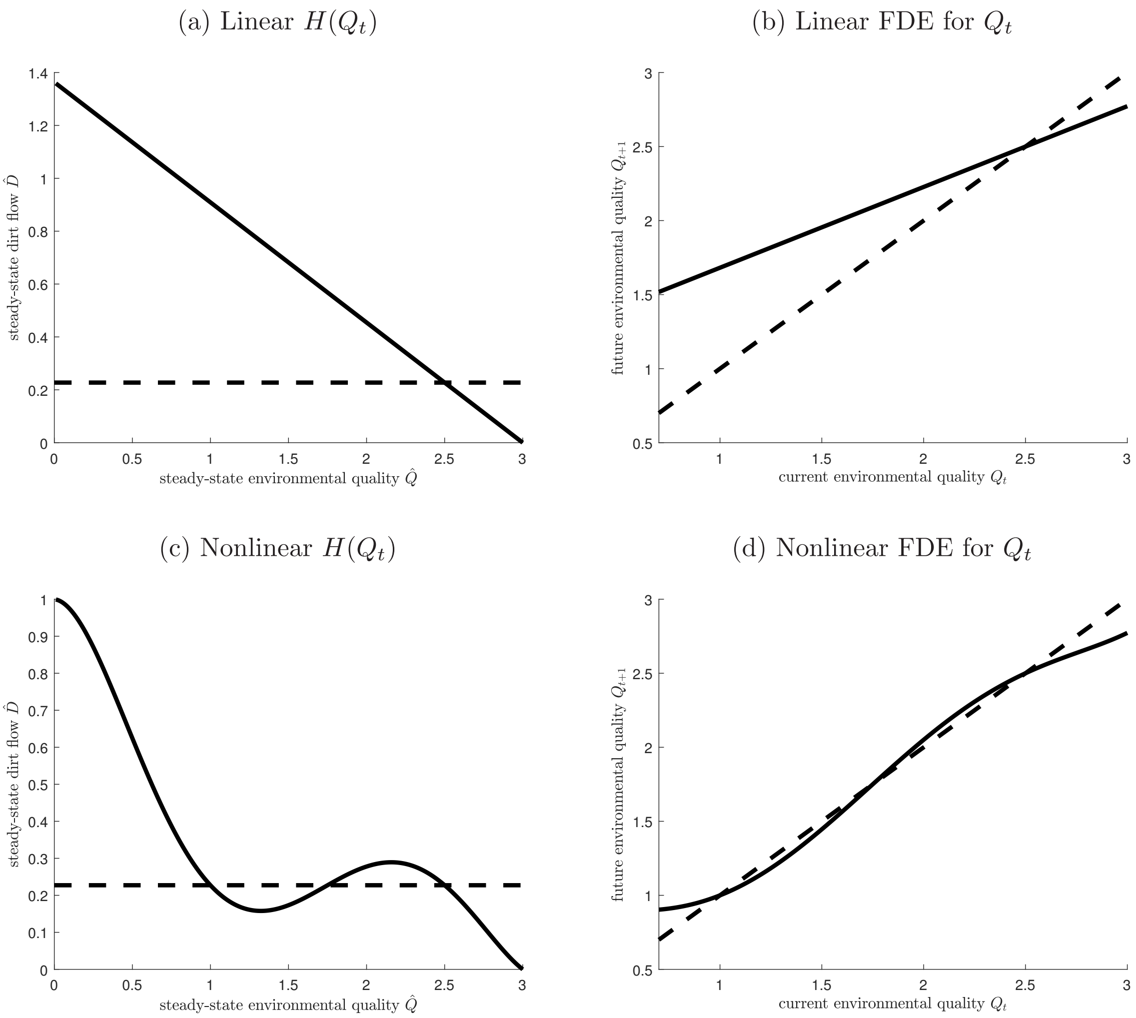


(f) Old-age consumption $\hat{c}^o$



Legend Steady-state environmental quality is such that $0 \leq \hat{Q} \leq \bar{Q}$. The values for $\hat{c}^o$, $\hat{c}^y$, $\hat{m}$, $\hat{k}$, and $\hat{D}$ are obtained by solving the system in (26)–(32) for all values of $\hat{Q}$ in the domain. Output satisfies $\hat{y} = \Omega\hat{k}^{\alpha}$.

Figure 3: Linear and non-linear $H(Q)$ functions



Note In the linear case the regeneration function is given by equation (33). The nonlinear case incorporates a quintic regeneration function as given in equation (35).

Table 3: Allocation and welfare

		(a) ME_c		(b) ME_d		(c) DSO_l		(d) DSO_n	
$\hat{Q}$	environmental quality	2.5000		1.0005		2.7604		2.7570	
$\hat{k}$	capital intensity	0.1643		0.1643		0.0642		0.0642	
$\hat{r}$	interest factor	1.0976		1.0979		2.7986		2.7986	
$\hat{r}^a$	annual interest rate (percent)	2.5000%		2.5005%		4.5492%		4.5492%	
$\hat{y}$	output per worker	1.0000		0.9999		0.7541		0.7541	
$\hat{w}$	wage rate	0.7000		0.6999		0.5279		0.5279	
$\hat{m}$	private abatement	0.2665	10^{-2}	0.2786	10^{-2}	1.5780	10^{-2}	1.5826	10^{-2}
$\hat{c}^y$	youth consumption	0.5330		0.5329		0.3248		0.3257	
$\hat{c}^o$	old-age consumption	0.3447		0.3447		0.3248		0.3257	
$\hat{g}$	public abatement	0.0000		0.0000		0.0420		0.0401	
$\hat{D}$	net dirt flow	0.2273		0.2270		0.1089		0.1106	
$\hat{\Lambda}^y$	life-time utility	6.0763		−0.9826		6.3352		6.3294	

Note With a linear environmental regeneration function $H(Q_t)$ the unmanaged market economy settles in the unique steady state labeled ME_c. If $H(Q_t)$ is nonlinear there is also a heavily polluted steady state for the unmanaged economy labeled ME_d. DSO_l and DSO_n denote the deterministic first-best social optimum for, respectively, the linear and nonlinear regeneration function.

For a given steady-state flow of dirt ($\hat{D}$), there exists a unique steady-state quality of the environment. The solid line in Figure 3(a) illustrates the relationship between $\hat{Q}$ and $\hat{D}$ for the linear case. Similarly, the solid line in Figure 3(b) depicts the fundamental difference equation for environmental quality, holding constant the total flow of dirt.

The key features of the steady-state market equilibrium are reported in Table 3(a). Environmental quality $\hat{Q}$ is (calibrated to be) close to its pristine level $\bar{Q}$ and we refer to this equilibrium as the clean steady state (ME$_c$). Private abatement is positive but rather small. Indeed, it is calibrated to be a half percent of youth consumption in the clean steady state. The characteristic roots of the linearized system (see (24)) equal $\lambda_1 = 0.2999$ and $\lambda_2 = 0.5454$ implying that the system is stable and converges to the unique steady steady state from any feasible initial condition (k_0, Q_0). In Figure 4 we illustrate the adjustment paths for k_t, Q_t, m_t, and D_t when the system faces the initial conditions $(0.1643, 1)$. At time $t = 0$ capital and environmental quality are predetermined. Private abatement is higher than its long-run level whilst the dirt flow is slightly lower than its steady-state level.

3.1.2 Non-linear environmental dynamics

In recent years prominent ecologists have argued that ecosystems may exhibit catastrophic shifts in the vicinity of threshold points (Scheffer *et al.*, 2001). Whilst such shifts are impossible when the regeneration function is linear (as in the previous subsection), they become possible when this function displays the right kind of non-linearity. In this subsection we study the dynamic behaviour of the unmanaged economy in the presence of tipping points.

To keep things simple we adopt the following cubic regeneration function:

$$H(Q_t) \equiv \phi_5 Q_t^5 + \phi_4 Q_t^4 + \phi_3 Q_t^3 + \phi_2 Q_t^2 + (1 + \phi_1) Q_t + \phi_0, \tag{35}$$

where the ϕ_i parameters are chosen such that the resulting fundamental difference equation for environmental quality is S-shaped and, for a given net dirt flow, features two stable steady states.[3] See Figure 3(d). The regeneration function itself has been illustrated in Figure 3(c) for different steady-state values of Q. The parameterization of $H(Q_t)$ is explained in detail in Appendix A.

By construction multiple equilibria are a key feature of the unmanaged market economy. Indeed, as is shown in Table 3(b), the dirty steady-state equilibrium (ME$_d$) is virtually identical to its clean counterpart (ME$_c$) except in terms of environmental quality which drops from $\hat{Q}_c = 2.5$ to $\hat{Q}_d = 1.0005$. Private abatement is slightly higher and the net dirt flow is slightly

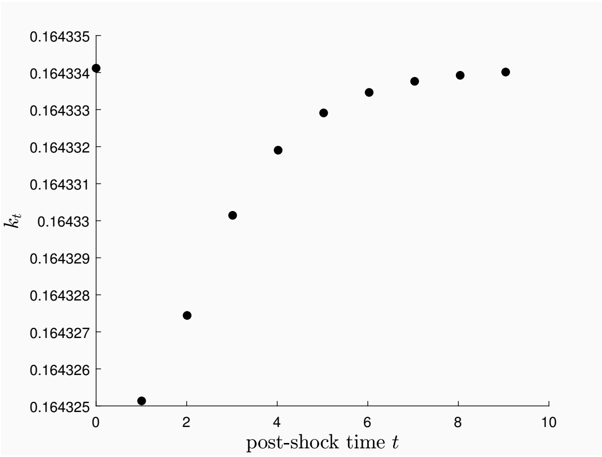
[3] In the literature on shallow lake dynamics a specific functional form of the regeneration function is typically employed which takes the following form:

$$P_{t+1} = (1 - \pi) P_t + \frac{P_t^2}{P_t^2 + 1} + D_t, \qquad \frac{1}{2} < \pi < \frac{3\sqrt{3}}{8},$$

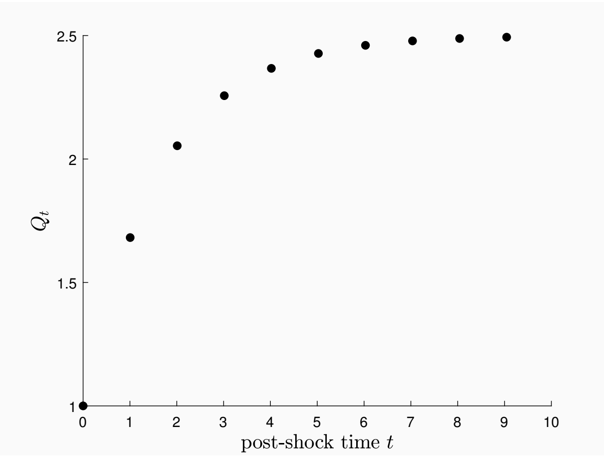
where P_t is the pollution stock at time t and $Q_t \equiv \bar{Q} - P_t$. This function is qualitatively similar to our quintic expression and we use the latter because it is easier to parameterize.

Figure 4: Transition to the unique steady state with a linear regeneration function

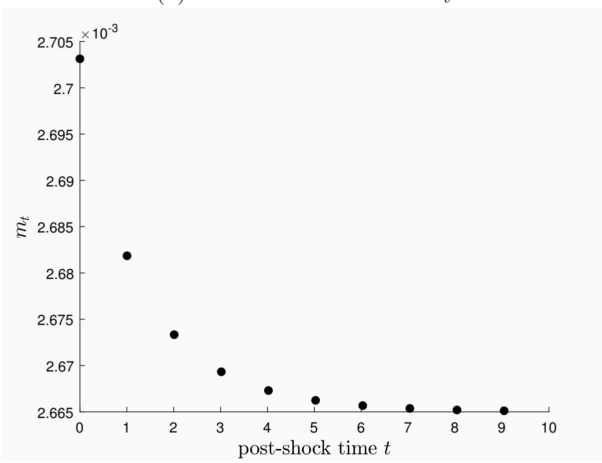
(a) Capital intensity k_t



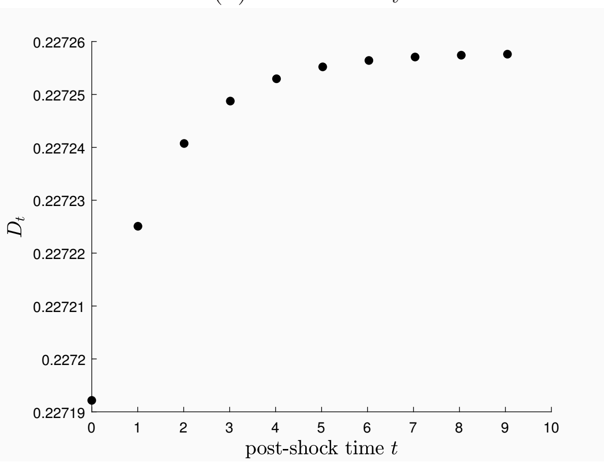
(b) Environmental quality Q_t



(c) Private abatement m_t



(d) Dirt flow D_t



Note The graphs plot the transitional dynamics in the different variables departing from the initial condition $(k_0, Q_0) = (0.1643, 1)$.

lower in ME_d than in ME_c. But these effects are not enough to cause a significant difference in the macroeconomic variables for the two equilibria. This is because $\hat{Q}_d$ lies far enough from the truly infernal region as shown in Figure 2 above. The characteristic roots for the two stable steady state equilibria are, respectively $(\lambda_1, \lambda_2) = (0.2999, 0.5454)$ for ME_d and $(\lambda_1, \lambda_2) = (0.2999, 0.6388)$ for ME_c. Hence both steady states are locally stable and the initial (k_0, Q_0) combination determines which equilibrium the system converges to.

3.2 Social optimum

In the unmanaged market equilibrium the government does not engage in abatement activities whilst individuals do. Since environmental quality is a non-excludable and non-rival public good, the clean market equilibrium is unlikely to be socially optimal. In this section we characterize the deterministic first-best social optimum (DSO hereafter) both with a linear and a nonlinear regeneration function.

In the presence of overlapping generations the social welfare function must take a specific form in order to yield a dynamically consistent social optimum. Specifically, as was stressed by Calvo and Obstfeld (1988), it is imperative that the old generation in the planning period is treated appropriately by applying reverse discounting. In the context of our model, the social welfare function is given by:

$$\begin{aligned} SW_t = {} & \frac{1}{\beta_G}\left[\ln c^y_{t-1} + \chi \ln m_{t-1} + \beta \ln c^o_t + \zeta\beta \ln Q_t\right] \\ & + \ln c^y_t + \chi \ln m_t + \beta \ln c^o_{t+1} + \zeta\beta \ln Q_{t+1} \\ & + \beta_G\left[\ln c^y_{t+1} + \chi \ln m_{t+1} + \beta \ln c^o_{t+2} + \zeta\beta \ln Q_{t+2}\right] + \ldots, \end{aligned} \tag{36}$$

where β_G is the social planner's discount factor $(0 < \beta_G < 1)$.[4] Note that we impose symmetry up-front and express social welfare per young person (worker), of which there are L. The key aspect guaranteeing dynamic consistency is that lifetime utility of the current old generation is 'blown up' by the inverse of the social discount factor. Of course, at time t the planner cannot influence the predetermined variables (c^y_{t-1}, m_{t-1}, and Q_t) but she can set the old-age consumption level c^o_t and reverse discounting ensures that this choice will be made consistently.

The equality constraints faced by the social planner are:

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c^y_t - c^o_t - m_t - g_t, \tag{37}$$

$$Q_{t+1} = H(Q_t) - D_t, \tag{38}$$

$$D_t = \xi L f(k_t) e^{\gamma L m_t - \eta L g_t}, \tag{39}$$

where $f(k_t) = \Omega k^\alpha_t$ is the intensive-form production function. In addition, the planner faces the

[4] In our formulation of the social welfare function we adopt the traditional approach by respecting each individual's preferences. As is pointed out by Andreoni (2006, p. 1224) the choice of how to treat warm-glow giving in social welfare is "as much a philosophical question as it is an economic one." Diamond (2006, pp. 909-910) and Andreoni (2006, p. 1227) propose excluding the warm-glow term in the social welfare function.

following inequality constraint:

$$g_t \geq 0. \tag{40}$$

Equation (37) is the resource constraint, (38) is the evolution equation for environmental quality, (39) defines the dirt flow, and (40) shows that public abatement must be non-negative.

At time t the predetermined variables are c^y_{t-1}, m_{t-1}, Q_t, and k_t and the choice variables of the planner are $c^y_{t+\tau}$, $c^o_{t+\tau}$, $m_{t+\tau}$, $Q_{t+1+\tau}$, $y_{t+\tau}$, $k_{t+1+\tau}$, $D_{t+\tau}$, and $g_{t+\tau}$ (for $\tau = 0, 1, \ldots$). We show the details of the derivations in Appendix B and focus here on the first-order conditions characterizing the interior solution for which public abatement is strictly positive. In addition to (37)–(39) they are:

$$\lambda^k_t = \frac{1}{c^y_t} = \frac{\beta}{\beta_G c^o_t} = \frac{\chi}{m_t} + \gamma L D_t \lambda^q_t, \tag{41}$$

$$\lambda^k_t = \beta_G \Big[\big(f'(k_{t+1}) + 1 - \delta\big) \lambda^k_{t+1} - \xi L f'(k_{t+1}) e^{-\gamma L m_{t+1} - \eta L g_{t+1}} \lambda^q_{t+1} \Big], \tag{42}$$

$$\lambda^q_t = \frac{\beta \zeta}{Q_{t+1}} + \beta_G H'(Q_{t+1}) \lambda^q_{t+1}, \tag{43}$$

$$\lambda^k_t = \eta L D_t \lambda^q_t, \tag{44}$$

where λ^k_t and λ^q_t are the shadow prices of capital and environmental quality respectively. For given initial conditions (k_t, Q_t), the perfect foresight solution selects unique time paths for k_{t+1}, Q_{t+1}, c^y_t, c^o_t, m_t, g_t, λ^k_t, and λ^q_t. In the remainder of this paper we assume that the social planner's discount factor coincides with the discount factor of individuals ($\beta_G = \beta$). The expressions in (41) imply that, in any given period, optimal consumption is the same for young and old individuals ($c^y_t = c^o_t$ for all t). The final task at hand is to numerically characterize the DSO for the linear and nonlinear regeneration functions.

3.2.1 Linear environmental dynamics

With the linear regeneration function as stated in (33) above, the steady-state equilibrium in the unmanaged economy is unique–see scenario ME$_c$ in Table 3(a). The key features of the DSO for this case have been reported in column (c) of that table. Even though the unmanaged market settles in a clean equilibrium, the DSO selects an even cleaner steady state than the market produces. It achieves this aim by (a) sharply reducing the capital intensity (and output per worker), (b) operating a sizeable program of public abatement (amounting to 5.58% of steady-state output), and (c) stimulating private abatement (which is almost six times higher in the DSO than in ME$_c$).

Figure 5 visualizes the transition from ME$_c$ to the first-best social optimum under a linear regeneration function. At the time of implementation of the policy initiative, the social planner proceeds at full throttle by selecting high values for both public and private abatement as well as consumption. This brings down the dirt flow and reduces the future capital intensity.

Environmental quality improves dramatically in the next period after which the abatement instruments are reduced substantially. Over time transition in the capital intensity is relatively fast and monotonic, whilst adjustments in environmental quality are also monotonic but somewhat slower.

3.2.2 Non-linear environmental dynamics

With the nonlinear regeneration function as stated in (35) above, there exist two steady-state equilibria in the unmanaged economy–a clean one (ME_c) and a dirty one (ME_d). See columns (a) and (b) in Table 3. Just as for the linear case studied above, the DSO is unique in the nonlinear case also–see the results for scenario DSO_n in Table 3(d). Comparing columns (c) and (d) we observe that the only slight differences occur in the values selected for Q, m, $c^y = c^o$, and D. These differences occur because both the level and the slope of the regeneration function differ at the social optimum between the linear and nonlinear cases.

4 Economic-environmental dynamics in a stochastic world

Up to this point we have followed standard practice in the literature by studying the economic-environmental dynamics in a deterministic world. In this section we broaden the horizon by moving to a stochastic setting. In particular, we assume that the difference equation for environmental quality is hit by random shocks in each period, i.e. equation (4) is replaced by:

$$Q_{t+1} = H(Q_t) - \phi_0 - D_t + \varepsilon_{t+1}, \tag{45}$$

where ε_{t+1} is drawn from a lognormal distribution with mean ϕ_0 and standard deviation ν, and $H(Q_t)$ is a cubic function as given in (35) above.[5] The random shock in the evolution equation for environmental quality ensures that young individuals are uncertain about the enjoyment they will derive from the environment when they are old. It follows that the relevant objective function of a young individual is his/her *expected* utility:

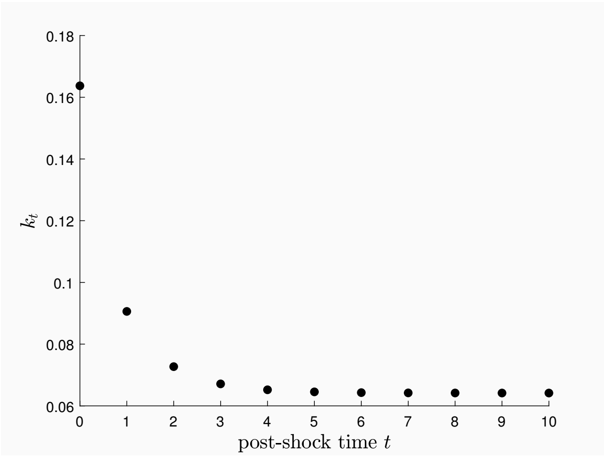
$$\mathbb{E}_t \left[\Lambda_t^i\right] \equiv \ln c_t^{y,i} + \chi \ln m_t^i + \beta \ln c_{t+1}^{o,i} + \beta\zeta \mathbb{E}_t \left[\ln Q_{t+1}\right], \tag{46}$$

where $\mathbb{E}_t [x]$ stands for the expectation of x, conditional on information available at time t. In the absence of further sources of randomness the conditional mean future environmental quality is $\mathbb{E}_t [Q_{t+1}] = H(Q_t) - D_t$. It follows that a given realization of ε_{t+1} has more impact on the individual's lifetime utility if $\mathbb{E}_t [Q_{t+1}]$ is low than if it is high.

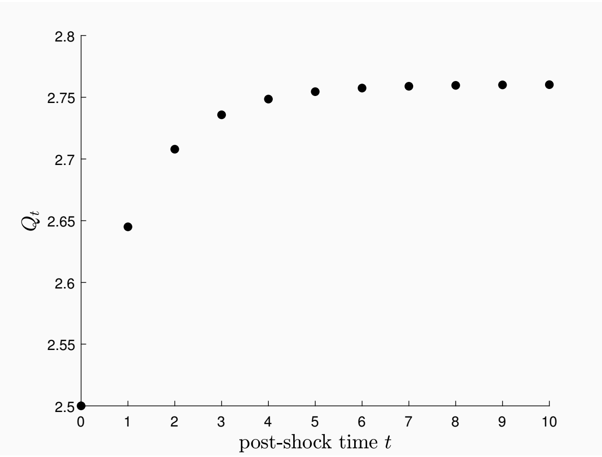
[5]To economize on space we restrict attention to the non-linear case in the main text. The case with a linear regeneration function is covered in Appendix C.

Figure 5: From the unmanaged economy to the first-best social optimum

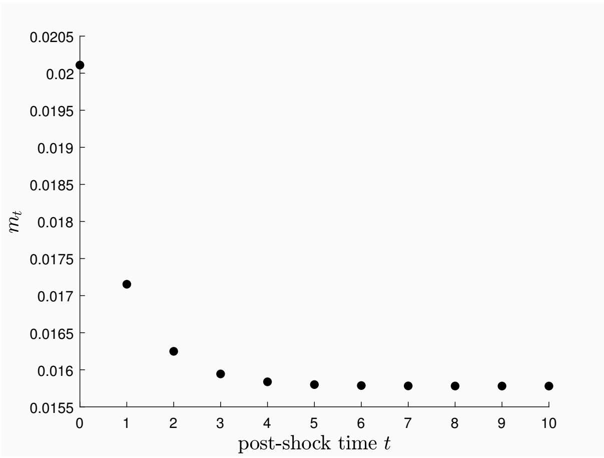
(a) Capital intensity k_t



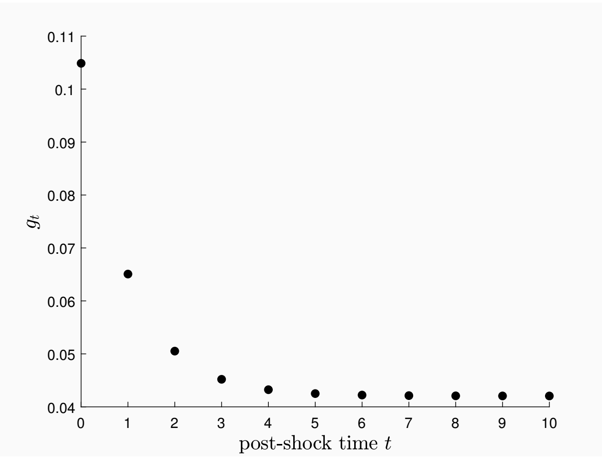
(b) Environmental quality Q_t



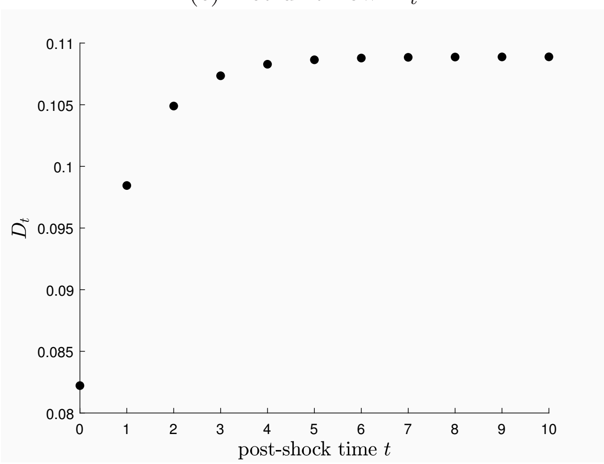
(c) Private abatement m_t



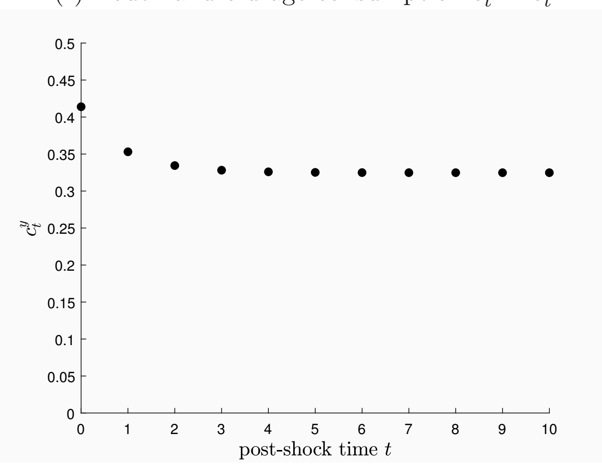
(d) Public abatement g_t



(e) Net dirt flow D_t



(f) Youth and old-age consumption $c_t^y = c_t^o$



4.1 Unmanaged market equilibrium

In the unmanaged market economy, government abatement and taxes are absent ($g_t = \tau_t = 0$), and young individual i chooses $c_t^{y,i}$, m_t^i, $c_{t+1}^{o,i}$, and s_t^i in order to maximize expected utility (46) subject to the lifetime budget constraint:

$$c_t^{y,i} + m_t^i + \frac{c_{t+1}^{o,i}}{1+r_{t+1}} = w_t, \tag{47}$$

and the environmental transition function (45). The individual takes as given the abatement expenditures by other individuals, $M_t^{-i} \equiv \sum_{j \neq i}^{L} m_t^j$. The first-order conditions consist of (47) and:

$$\lambda_t = \frac{1}{c_t^{y,i}} = \frac{\beta(1+r_{t+1})}{c_{t+1}^{o,i}} = \frac{\chi}{m_t^i} + \beta\zeta \frac{\partial \mathbb{E}_t \left[\ln Q_{t+1}\right]}{\partial m_t^i}, \tag{48}$$

where (5) and (45) imply that:

$$\frac{\partial \mathbb{E}_t \left[\ln Q_{t+1}\right]}{\partial m_t^i} = \mathbb{E}_t \left[\frac{\gamma \xi L y_t e^{-\gamma(m_t^i + M_t^{-i})}}{H(Q_t) - \phi_0 - \xi L y_t e^{-\gamma(m_t^i + M_t^{-i})} + \varepsilon_{t+1}} \right]. \tag{49}$$

By invoking symmetry and recognizing the dependence of output and the wage rate on the capital intensity we find that the unmanaged market equilibrium is the solution to:

$$\frac{\chi}{m_t} + \beta\zeta \mathcal{M}(m_t, k_t, Q_t) = \frac{1+\beta}{(1-\alpha)\Omega k_t^{\alpha} - m_t}, \tag{50}$$

$$c_t^y \equiv \frac{(1-\alpha)\Omega k_t^{\alpha} - m_t}{1+\beta}, \tag{51}$$

$$k_{t+1} = (1-\alpha)\Omega k_t^{\alpha} - m_t - c_t^y, \tag{52}$$

$$Q_{t+1} = H(Q_t) - \phi_0 - \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t} + \varepsilon_{t+1}, \tag{53}$$

where $\mathcal{M}(m_t, k_t, Q_t)$ is an auxiliary function defined as:

$$\mathcal{M}(m_t, k_t, Q_t) \equiv \mathbb{E}_t \left[\frac{\gamma \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t}}{H(Q_t) - \phi_0 - \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t} + \varepsilon_{t+1}} \right]. \tag{54}$$

Equation (50) is a rewritten version of the first-order condition for private abatement using the expression for youth consumption, conditional on private abatement, as stated in (51). Equations (52) and (53) state the dynamic evolution of, respectively, the capital intensity and environmental quality conditional on private abatement.

The unmanaged market model is solved as follows. First, for given values of k_t and Q_t equation (50) is solved for optimal private abatement by simulating lognormally distributed random variables and conducting quasi Monte Carlo integration to compute the $\mathcal{M}(m_t, k_t, Q_t)$ function. This yields the 'policy' function $m_t = \mathbf{m}(k_t, Q_t)$. Second, by substituting $\mathbf{m}(k_t, Q_t)$

into (51)–(53) we obtain the policy functions for c_t^y, k_{t+1}, and Q_{t+1}:

$$\mathbf{c}_t^y(k_t, Q_t) \equiv \frac{(1-\alpha)\Omega k_t^\alpha - \mathbf{m}(k_t, Q_t)}{1+\beta}, \tag{55}$$

$$\mathbf{k}^+(k_t, Q_t) \equiv (1-\alpha)\Omega k_t^\alpha - \mathbf{m}(k_t, Q_t) - \mathbf{c}_t^y(k_t, Q_t), \tag{56}$$

$$\mathbf{Q}^+(k_t, Q_t) \equiv H(Q_t) - \phi_0 - \xi L \Omega k_t^\alpha e^{-\gamma L \mathbf{m}(k_t, Q_t)}, \tag{57}$$

where next period's capital intensity is deterministic, as $k_{t+1} \equiv \mathbf{k}^+(k_t, Q_t)$, and future environmental quality is stochastic, because $Q_{t+1} = \mathbf{Q}^+(k_t, Q_t) + \varepsilon_{t+1}$.

In Figure 6 we illustrate simulated time paths for the main variables in the model as predicted by our model. We adopt a very long-run perspective by simulating one thousand periods. Since each period represents thirty years this amounts to thirty thousand years. Details concerning the computational aspects for these simulations are found in Appendix C. The key features of these simulations are as follows. First, environmental quality displays distinct and often long-lived epochs during which it is stuck fluctuating around either the clean or the dirty equilibrium. An unfortunate sequence of bad draws for ε_{t+1} can push the system into the basin of attraction consistent with the polluted (low-welfare) stochastic equilibrium path. Since the government does not manage the economy, only a sequence of advantageous draws for ε_{t+1} can get the system out of this trap. Second, whilst fluctuations in the capital intensity are rather small, they are asymmetric in the sense that there is an upper bound beyond which capital does not move. Intuitively this is because youth consumption is essential so that savings will always fall well short of wages. The asymmetric pattern of fluctuations are also found in private abatement, youth- and old-age consumption, and the net dirt flow. Third, private abatement is considerably higher during high pollution epochs.

Since individuals care for the environment they do try to get out of a bad equilibrium by means of private abatement. To illustrate this mechanism further, Figure 7 provides some visual information on the situation within a given high-pollution era. As is clear from panel (c) private abatement is higher than usual in the polluted equilibrium but it is not sufficiently high to quickly return the system to a clean epoch–the warm glow effect is not powerful enough to do so.

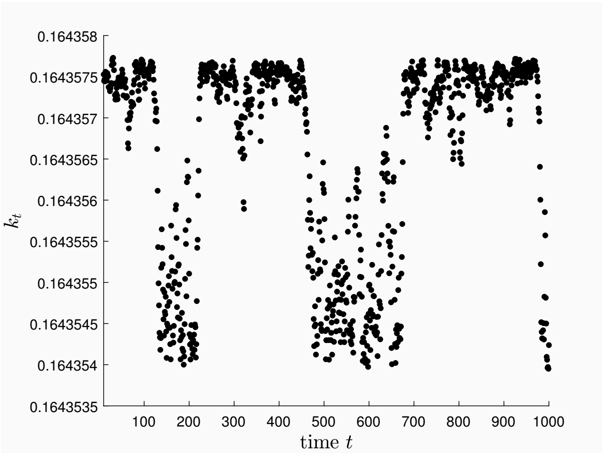
4.2 Social optimum

Like individuals agents the social planner is unable to observe future environmental shocks. As a result the social welfare function as stated in (36) is augmented in a stochastic setting to:

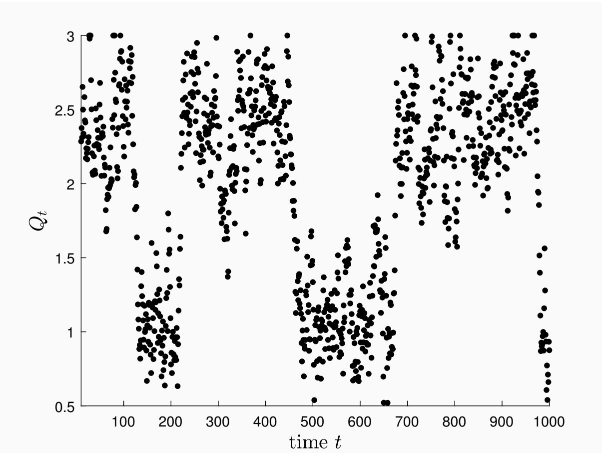
$$\begin{aligned} \mathbb{E}_t\left[SW_t\right] &\equiv \mathbb{E}_t \sum_{\tau=0}^{\infty} \beta_G^{\tau-1}\left[\ln c_{t+\tau-1}^y + \chi \ln m_{t+\tau-1} + \beta \ln c_{t+\tau}^o + \zeta\beta \ln Q_{t+\tau}\right] \\ &= \frac{1}{\beta_G}\left[\ln c_{t-1}^y + \chi \ln m_{t-1}\right] + \mathbb{E}_t \sum_{\tau=0}^{\infty} SF(c_{t+\tau}^y, m_{t+\tau}, c_{t+\tau}^o, Q_{t+\tau})\beta_G^\tau, \end{aligned} \tag{58}$$

Figure 6: The unmanaged economy in a stochastic world: long-run view

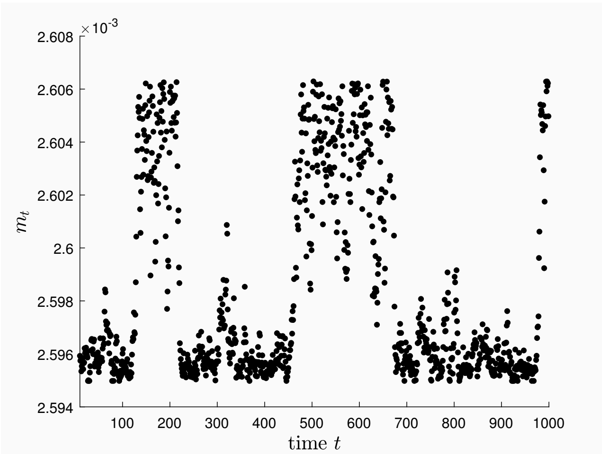
(a) Capital intensity k_t



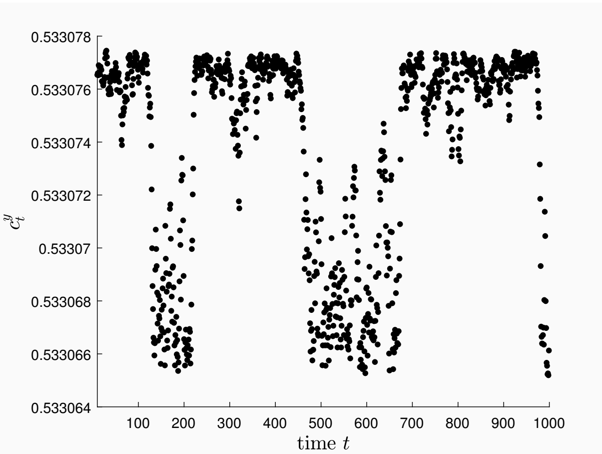
(b) Environmental quality Q_t



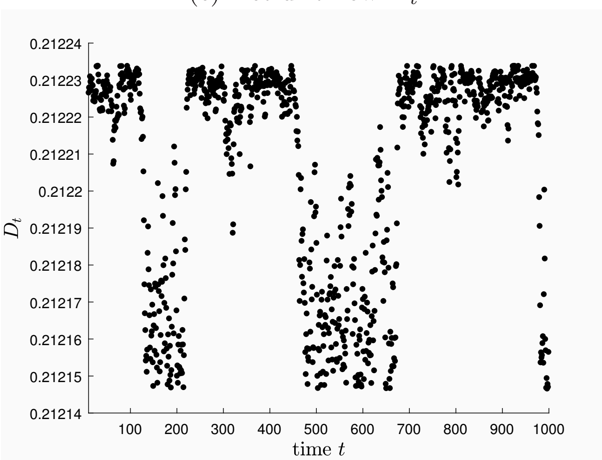
(c) Private abatement m_t



(d) Youth consumption c_t^y



(e) Net dirt flow D_t



(f) Old-age consumption c_t^o

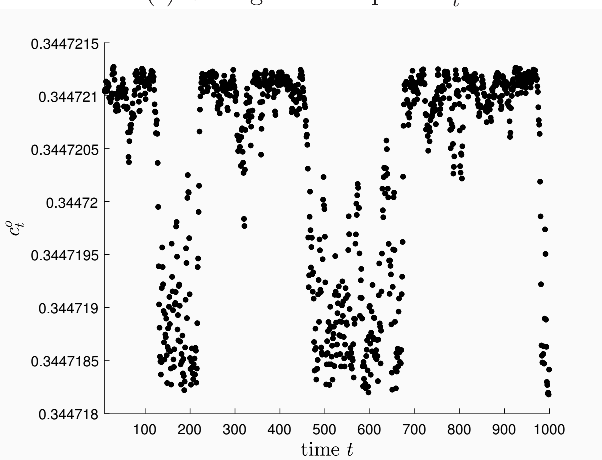
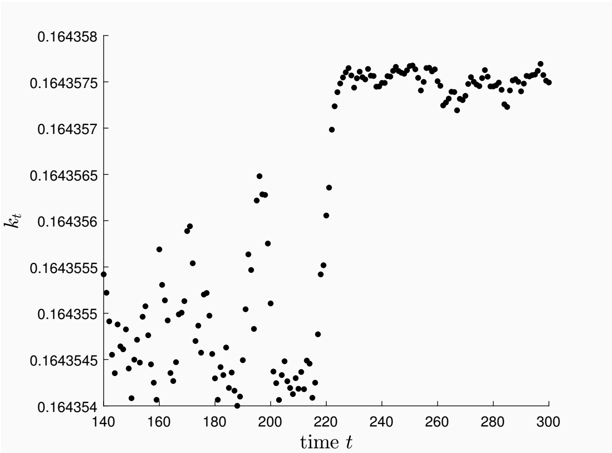
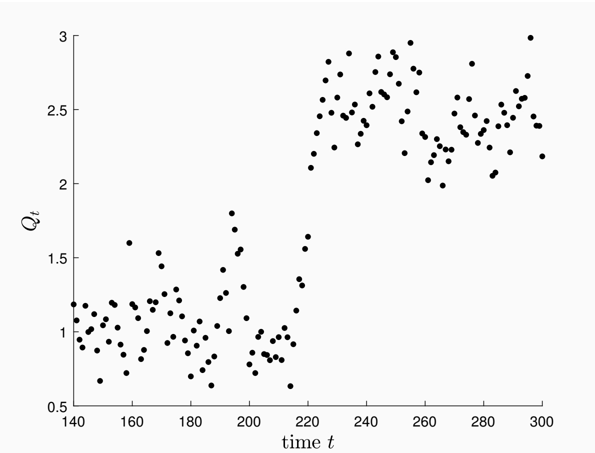


Figure 7: The unmanaged economy in a stochastic world: bad times

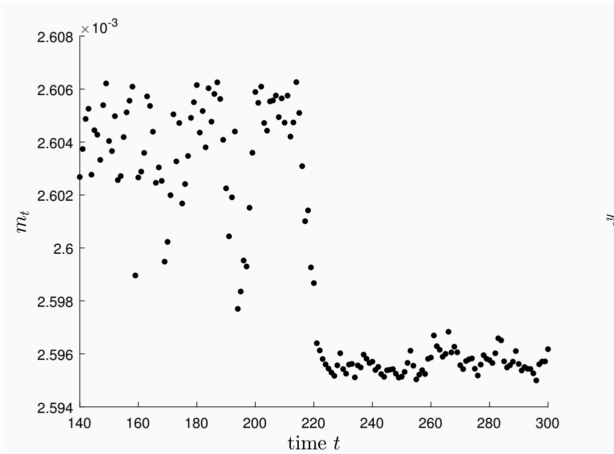
(a) Capital intensity k_t



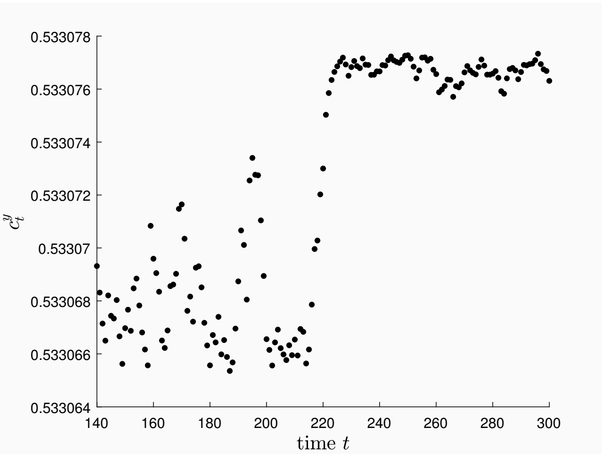
(b) Environmental quality Q_t



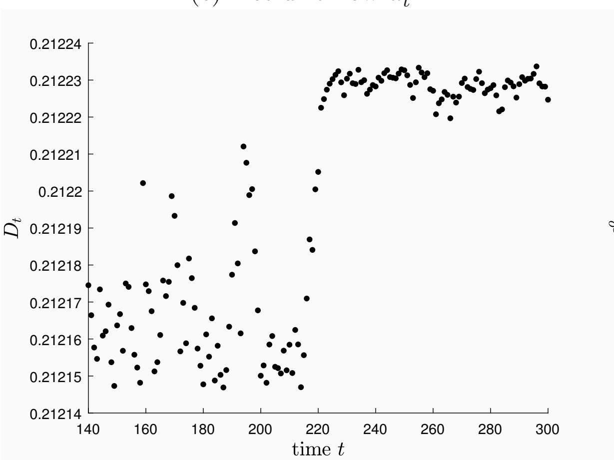
(c) Private abatement m_t



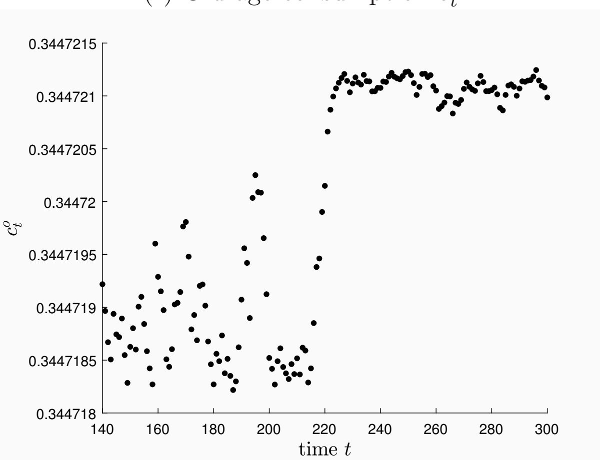
(d) Youth consumption c_t^y



(e) Net dirt flow d_t^n



(f) Old-age consumption c_t^o



where the social felicity function is defined as follows:

$$SF(c^y_{t+\tau}, m_{t+\tau}, c^o_{t+\tau}, Q_{t+\tau}) \equiv \ln c^y_{t+\tau} + \chi \ln m_{t+\tau} + \frac{\beta}{\beta_G}\left[\ln c^0_{t+\tau} + \zeta \ln Q_{t+\tau}\right]. \tag{59}$$

Note that $SF(\cdot)$ differs from an individual's felicity function $\Lambda^i_t(\cdot)$ as stated in (1) above because the former contains only variables affecting individuals living in the same time period whereas the latter expresses lifetime utility over the life cycle of an individual and thus contains variables in adjacent periods.

By defining augmented social welfare as:

$$ASW_t \equiv SW_t - \frac{1}{\beta_G}\left[\ln c^y_{t-1} + \chi \ln m_{t-1}\right], \tag{60}$$

we find that ASW_t can be written in a recursive format:

$$ASW_t(k_t, Q_t) = SF(c^y_t, m_t, c^o_t, Q_t) + \beta_G \mathbb{E}_t\left[ASW_{t+1}(k_{t+1}, Q_{t+1})\right]. \tag{61}$$

The first-best social optimum can now be described using the tools of dynamic programming. The Bellman equation is given by:

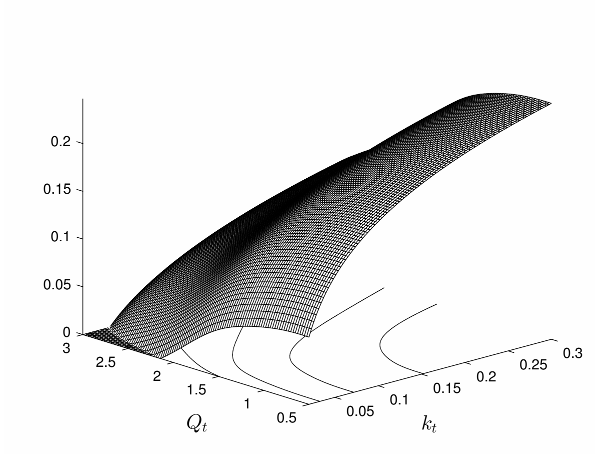
$$\begin{aligned} \mathcal{V}(k_t, Q_t) = & \max_{\{c^y_t, m_t, c^o_t, g_t\}} SF(c^y_t, m_t, c^o_t, Q_t) + \beta_G \mathbb{E}_t\left[\mathcal{V}(k_{t+1}, Q_{t+1})\right] \\ \text{s.t.} \quad & k_{t+1} = f(k_t) + (1-\delta)k_t - c^y_t - c^o_t - m_t - g_t \\ & Q_{t+1} = H(Q_t) - \phi_0 - D_t + \varepsilon_{t+1} \\ & D_t \equiv \xi L f(k_t) e^{-\gamma L m_t - \eta L g_t} \\ & g_t \geq 0. \end{aligned} \tag{62}$$

where c^y_t, m_t, c^o_t, and g_t are the control variables, k_t and Q_t are the state variables, and $\mathcal{V}(k_t, Q_t)$ is the value function.

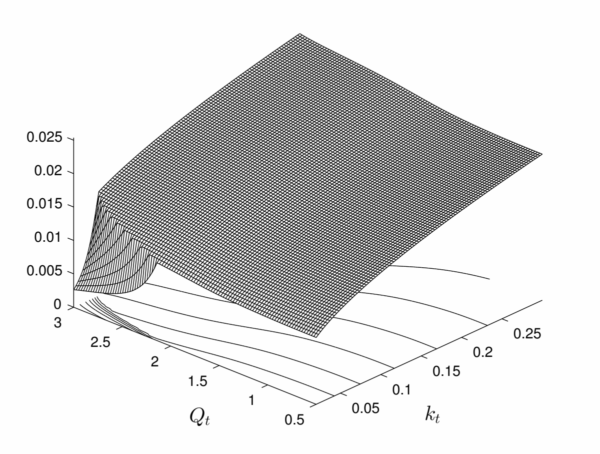
In Figure 8 we illustrate the policy functions for public abatement $g_t = \mathbf{g}(k_t, Q_t)$, private abatement $m_t = \mathbf{m}(k_t, Q_t)$, consumption $c^y_t = c^o_t = \mathbf{c}(k_t, Q_t)$, the future capital intensity $k_{t+1} = \mathbf{k}^+(k_t, Q_t)$, and the deterministic part of next period's environmental quality $Q_{t+1} - \varepsilon_{t+1} = \mathbf{Q}^+(k_t, Q_t)$. Details concerning the numerical approach adopted to compute these policy functions are found in Appendix C. The key features of these policy functions are as follows. First, for a given capital intensity public abatement (panel (a)) is decreasing in the quality of the environment and even becomes zero when Q_t is high and k_t is low. Second, for a given environmental quality public abatement is increasing in the capital intensity. Third, for a given capital intensity private abatement (panel (b)) is generally increasing in the quality of the environment. In a rich world blessed with a clean environment (k_t and Q_t both high), the social planner finds it optimal to let private individuals engage in a relatively high level of private abatement. Matters are different when the capital intensity is low. In such a setting optimal private abatement is non-monotonic and becomes a downward sloping function of Q_t beyond

Figure 8: Policy functions in the SSO

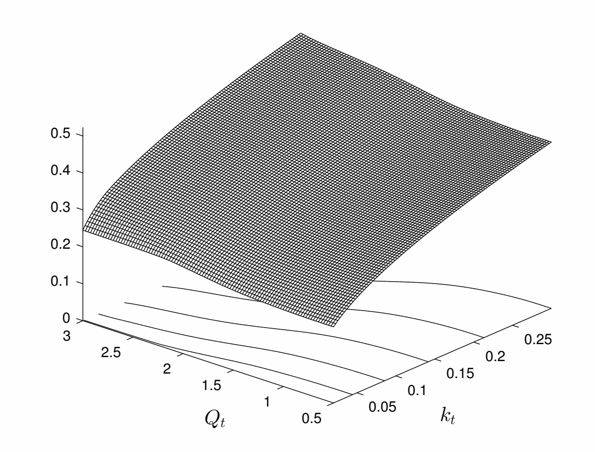
(a) Public abatement $\mathbf{g}(k_t, Q_t)$



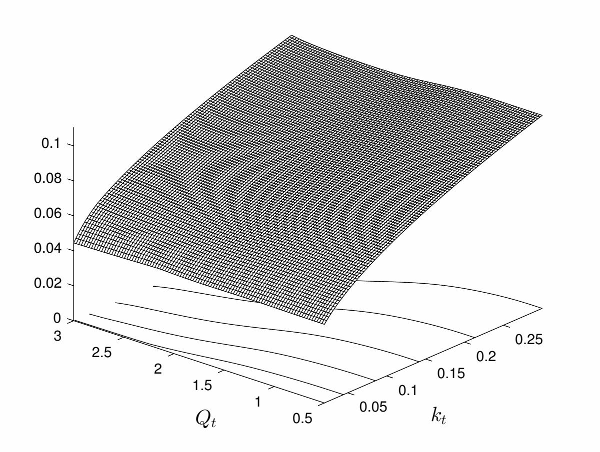
(b) Private abatement $\mathbf{m}(k_t, Q_t)$



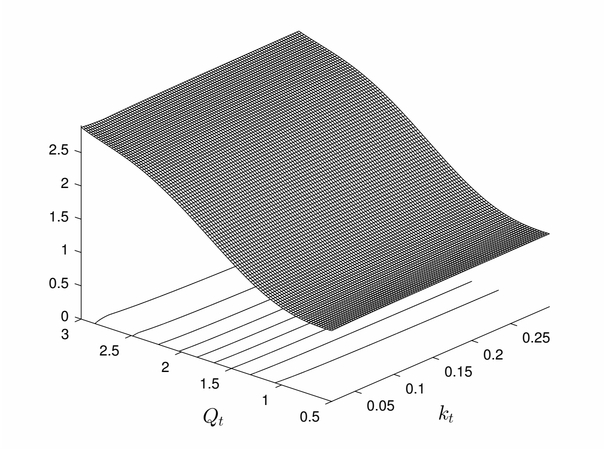
(c) Consumption $\mathbf{c}(k_t, Q_t)$



(d) Future capital intensity $\mathbf{k}^+(k_t, Q_t)$



(e) Planned future environmental quality $\mathbf{Q}^+(k_t, Q_t)$



a high enough level of environmental quality. Fourth, optimal consumption (during youth and old-age) is decreasing in environmental quality and strongly increasing in k_t. Intuitively, in a polluted world the social planner finds it more important to spend resources on public abatement than on consumption or saving (compare panels (a), (c), and (d)). Fifth, for a given level of Q_t the deterministic part of future environmental quality is virtually independent of the capital intensity.

In order to demonstrate the long-run statistical properties of the economic-ecological system run by a social planner we simulate the model for $T = 10^5$ periods and use a kernel estimation method to compute the resulting probability density functions for the different choice variables. In Figure 9(a) the PDF for public abatement is bell-shaped with g_t ranging from its lower bound ($g_t = 0$) to about $g_t = 0.07$. A social planner will thus conduct public abatement almost all of the time. In doing so the planner ensures that environmental quality will be high almost all the time, ranging from $Q_t \approx 2.1$ to its upper bound ($Q_t = \bar{Q}$) as is shown in panel (d) of Figure 9. Despite the fact that the fundamental difference equation for environmental quality features two (deterministic) steady states, the policy maker will ensure that fluctuations take place around the clean steady state. Panel (b) of Figure 9 shows that there is substantial variability in the optimal level of private abatement. Finally, panel (c) shows that the capital intensity has a multi-modal PDF with a relatively tight support.

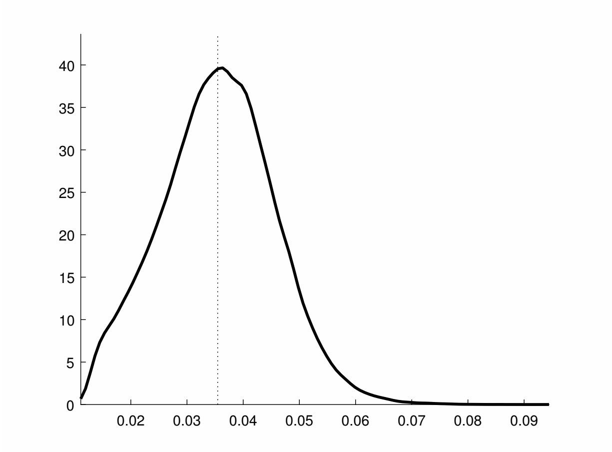
4.3 The veil of ignorance

Up to this point in the paper we have restricted attention to two extreme scenarios, namely the messy and completely unmanaged laissez faire situation and the cerebral world of a benevolent social planner implementing the stochastic first-best social optimum (SSO). In a world hit by stochastic shocks there will be fluctuations in all the variables of interest to the individuals populating the economy. This prompts the following question. Given that stochastic fluctuation are a fact of life under both scenarios, in which world would you like to be born if you do not know the (k_t, Q_t) combination that you will face at birth? Would you like to live in the turbulent world of the market economy or would you prefer the system managed by a social planner? To provide some perspective on this question we present Figure 10 which depicts the PDFs for environmental quality, Q_t, and expected lifetime utility at birth, $\mathbb{E}_t\left[\Lambda_t^y(k_t, Q_t)\right]$, resulting in the market equilibrium (solid lines) and in the first-best social optimum (dashed lines). We recall from the preceding discussion that the two cases differ in two important dimensions. First, there is no public abatement in the market economy whereas such abatement is set optimally in the SSO. Second, in the market economy old-age consumption is determined by the individual agent during youth in the market economy. In contrast, in the SSO the social planner determines old-age consumption without any regard to what youth consumption was the period before.

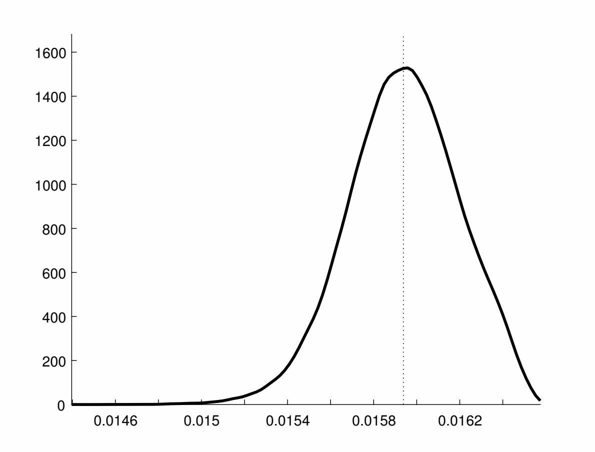
The key features of Figure 10 are as follows. First, in the unmanaged market economy the PDFs for environmental quality and expected utility at birth are both multimodal. This is, of course, consistent with the epochs that are clearly visible in the simulated time series displayed in Figure 6(b). Second, in the unmanaged market economy the supports of the distributions are

Figure 9: Probability density functions in the SSO

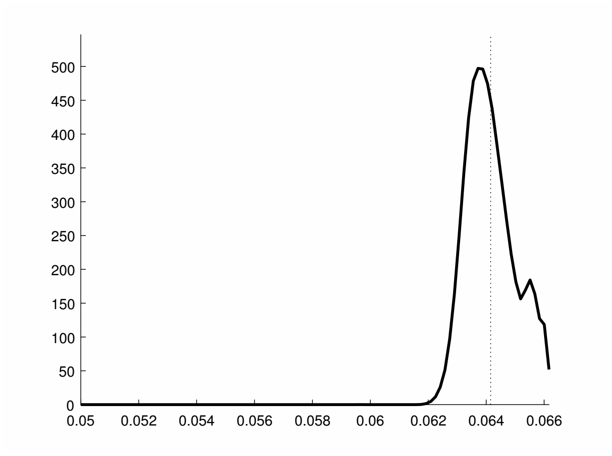
(a) Public abatement g_t



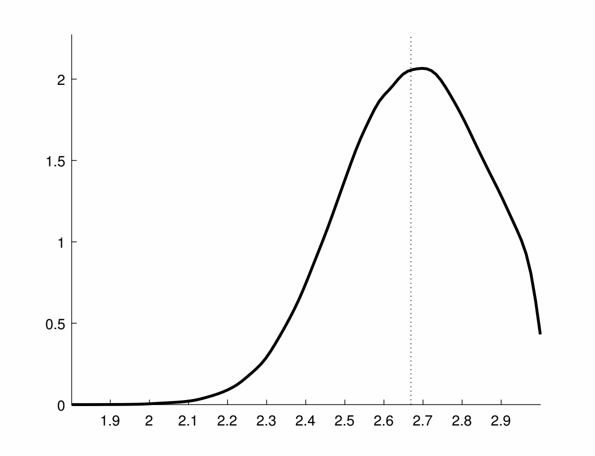
(b) Private abatement m_t



(c) Capital k_t



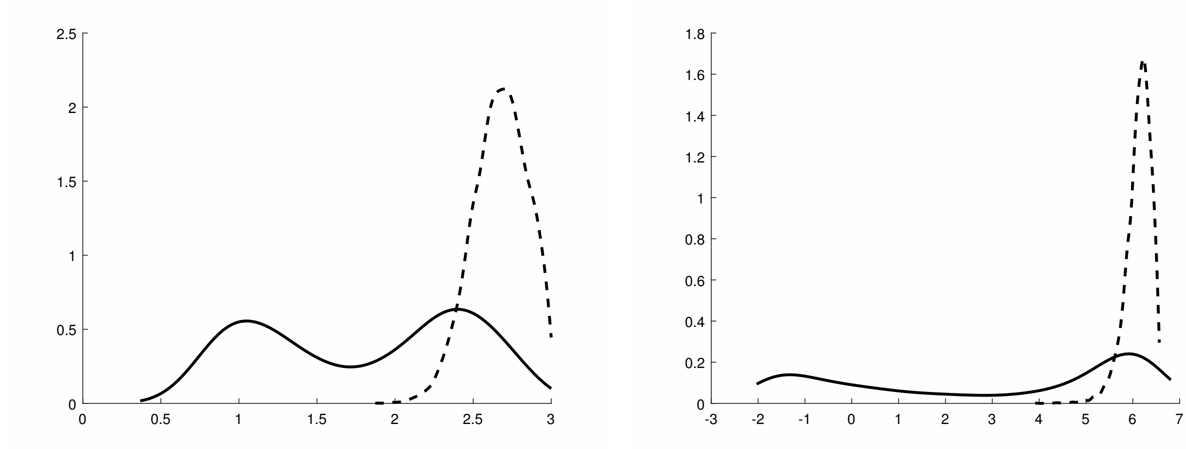
(d) Environmental quality Q_t



Legend The figures plot the estimated probability density functions (PDF) for a number of key variables. The thin dashed lines represent the sample averages for these variables.

Figure 10: The market or the planner?

(a) Environmental quality Q_t (b) Expected lifetime utility at birth $\mathbb{E}_t\left[\Lambda_t^y(k_t, Q_t)\right]$



quite wide, i.e. there exists a lot of inequality between generations and in that sense it matters a lot whether one is born during a clean epoch or in a polluted one. Third, in the SSO both distributions are single-peaked and their supports are relatively tight. Indeed, we find that there is absolutely no chance at all to be born in a polluted epoch in the SSO. In essence the social planner eliminates the polluted stochastic steady state by means of its abatement policy. Fourth, as is clear from Figure 10(b) there is a small probability that a given ('very lucky') generation is better off under the unmanaged market economy than under the SSO. This unlikely event can occur if environmental quality is very close to its paradisiacal level ($Q_t \approx \bar{Q}$). Since the social planner pursues intergenerational redistribution (in order to ensure that the co-existing old and young generations have the same consumption level) and the unmanaged market does not, a non-altruistic individual is better of in the unmanaged market economy in that specific case.

4.4 An ad hoc policy rule

As is clear from the policy functions depicted in Figure 8, the decentralization of the SSO is far from trivial. Indeed, in addition to calling for the optimal amount of public abatement (panel (a)), the SSO also aims to achieve the optimal distribution of resources between generations (such that $c_t^y = c_t^o$ for all t in panel (c)) *and* to engineer the socially optimal amount of private abatement (panel (b)). As a result it may simply not be possible to implement the SSO in its full complexity in actual economies. For example, which instrument would the planner use to induce private individuals to pursue the correct amount of private abatement?

In this section we investigate whether and to what extent an *ad hoc policy rule* for public abatement could approximate the welfare gains that are achievable under the SSO. As we have argued above, both the capital intensity and private abatement have rather limited effects on the environment. This suggests that a policy rule for public abatement only should perform adequately.

The construction of the ad hoc rule proceeds as follows. First we note from Figure 6(a) that fluctuations in the capital intensity are very small for the unmanaged market economy. Second, by setting k_t equal to its long-run average ($\bar{k} = 0.165$) and taking a slice out of the surface of the policy function for public abatement in Figure 8(a) we find the dashed line in Figure 11(c). The single-dimensional (average) policy function for public abatement is downward sloping and non-linear. Since environmental quality does not fall below $Q_t = 0.5$ in the unmanaged market economy (see Figure 6(a)) we restrict the domain for the policy function to the interval $[0.5, \bar{Q}]$. Third, by fitting a straight line though this average policy function we find the solid line in Figure 11(c). To summarize, the ad hoc policy rule is given by:

$$g_t = \pi_0 - \pi_1 Q_t, \tag{63}$$

for $Q_t \in [0.5, \bar{Q}]$ and with $\pi_0 = 0.2601$ and $\pi_1 = -0.0616$. Other than setting public abatement according to equation (63) the policy maker does not interfere in the economy or the environment.

In Figure 11, panels (a) and (b) we compare the PDFs of the SSO (dashed lines) to the ones attained under the ad hoc public abatement rule (solid lines). Interestingly the distributions for environmental quality differ only slightly so in that dimension the ad hoc rules performs quite well–see panel (a). For expected utility at birth the ad hoc rule performs well in the sense that environmental catastrophes are avoided with probability one. But the ad hoc rule performs somewhat worse than the first-best policy function in the sense that the distribution is shifted to the left. Intuitively, the ad hoc rule does not incorporate intergenerational risk sharing and does not produce the optimal amount of private abatement. This results in unrealized welfare gains.

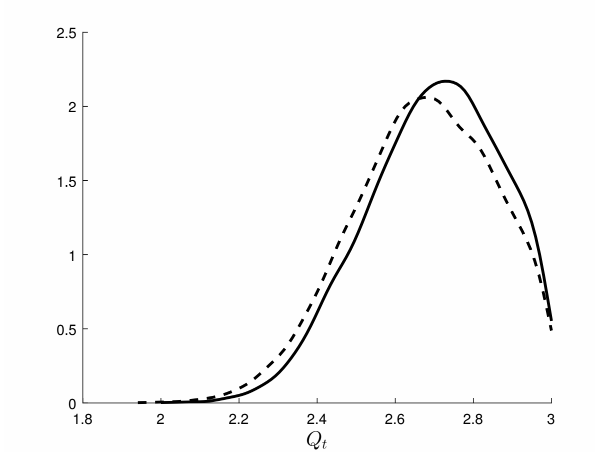
5 Conclusions

In this paper we have studied the interactions between the environment and the macroeconomic system employing a stochastic overlapping generations model of the Diamond-Samuelson type. In the absence of government policies and with a nonlinear environmental regeneration function, the stochastic unmanaged market economy displays often long-lasting epochs during which environmental quality remains very high whilst at other times the ecological system is trapped fluctuating around a highly polluted equilibrium. Even though individuals care for the environment they are unable to avoid such low-welfare epochs thus opening up a useful role for government intervention.

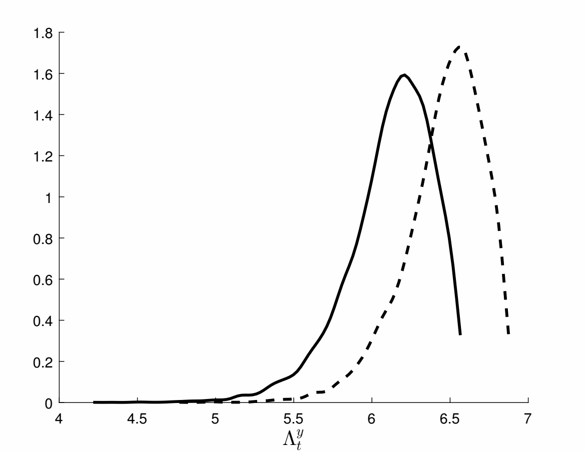
A dynamically consistent social planner operating with the same information set as the public will ensure that the low-quality trap is eliminated altogether. In the social optimum the policy maker conditions the allocation at each time on the pre-existing capital intensity and environmental quality. Since we assume that public abatement is more efficient than private

Figure 11: Performance of an ad hoc public abatement policy rule

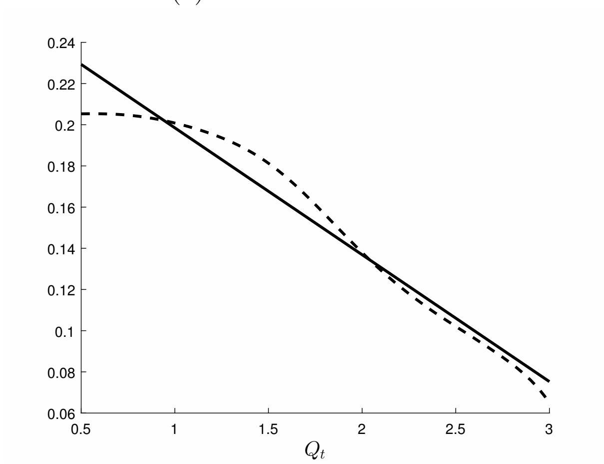
(a) Environmental quality Q_t



(b) Expected lifetime utility at birth $\mathbb{E}_t\left[\Lambda_t^y\right]$



(c) Public abatement



abatement, the policy function for public abatement (evaluated for the average capital intensity) is strongly decreasing in pre-existing environmental quality whilst the one for private abatement displays the opposite pattern. Intuitively, if environmental quality is very low then it is optimal for the government to conduct abatement. In contrast, if environmental quality is high then the marginal gains due to public abatement are low and it is advantageous to let individual agents engage in more private abatement for which they gain direct utility due to the warm-glow motive. Surprisingly, an ad hoc linear rule for public abatement which is only conditioned on the pre-existing quality of the environment captures most of the benefits attained under the first-best policy.

In this paper we have deliberately restricted attention to stochastic shocks affecting the environmental state equation. In future work we intend to introduce additional randomness in the form of productivity shocks affecting the economic system. Such shocks should have a non-trivial influence on optimal environmental policy. Indeed, on the one hand a positive productivity shock increases output and wages (which enhances welfare) but on the other hand it also increases the pollution inflow (which reduces welfare). We expect to find that part of the increase in wages will be used to increase public abatement in the first-best social optimum.

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Appendix A: Parameterization

A.1 Parameterizing the linear deterministic market equilibrium

In this appendix we document how the structural parameters in Table 2 were chosen.

- Principles of the steady state:
  - Each life phase lasts thirty years: $N = 30$.
  - Each cohort consists of one hundred people: $L = 100$.
  - There is no public abatement: $g_t = 0$.
  - The environment is in the clean deterministic steady state consistent with the linear regeneration function given in equation (33).
- The fixed parameters are:
  - Efficiency parameter of capital: $\alpha = 0.3$
  - The annual environmental regeneration rate is $\theta_a = 0.02$ which implies:

$$\theta = 1 - (1 - \theta_a)^N = 0.4545.$$

  - The private annual rate of time preference is $\rho_a = 0.06$ which gives:

$$\rho = (1 + \rho_a)^N - 1 = 2.2434, \quad \text{and} \quad \beta \equiv \frac{1}{1+\rho} = 0.3083.$$

  - Maximum environmental quality is $\bar{Q} = 3$
  - The weight attached to future environmental quality in the utility function: $\zeta = 25$.
- The parameterization targets are:
  - Target output level is $\hat{y} = 1$.
  - Target environmental quality in the clean steady state is $\hat{Q} = 2.5$.
  - Annual real interest rate is $\hat{r}_a = 0.025$ which gives the interest factor:

$$\hat{r} = (1 + \hat{r}_a)^N - 1 = 1.0976.$$

  - Target private abatement-consumption ratio:

$$\phi_{mc} \equiv \frac{\hat{m}}{\hat{c}^y} = 0.005.$$

  - Target relative contribution of private abatement to clean-up (in terms of dirt), $\phi_{\gamma md}$:

$$\begin{aligned} \phi_{\gamma md} &\equiv 1 - e^{-\gamma L \hat{m}} = 0.02, \\ \hat{d} &= \xi \hat{y} \left(1 - \phi_{\gamma md}\right). \end{aligned}$$

– Target relative efficiency of private versus public abatement:

$$\phi_{\gamma\eta} \equiv \frac{\gamma}{\eta} = 0.90.$$

- The steady state is fully characterized by:

$$\hat{c}^o = \beta(1+\hat{r})\hat{c}^y, \tag{A.1}$$

$$\hat{c}^o = (1+\hat{r})\hat{k}, \tag{A.2}$$

$$\hat{w} = \hat{c}^y + \frac{\hat{c}^o}{1+\hat{r}} + \hat{m}, \tag{A.3}$$

$$\frac{1}{\hat{c}^y} = \frac{\chi}{\hat{m}} + \frac{\beta\gamma\zeta\hat{D}}{\hat{Q}}, \tag{A.4}$$

$$\hat{r} + \delta = \frac{\alpha\hat{y}}{\hat{k}}, \tag{A.5}$$

$$\hat{w} = (1-\alpha)\hat{y}, \tag{A.6}$$

$$\hat{y} = \Omega\hat{k}^\alpha, \tag{A.7}$$

$$\hat{D} = \xi L\hat{y}e^{-\gamma L\hat{m}}, \tag{A.8}$$

$$\hat{Q} = \bar{Q} - \frac{1}{\theta}\hat{D}. \tag{A.9}$$

- Parameterization of the system proceeds as follows.

  – Set $\hat{Q} = 2.5$ and use (A.9) to compute $\hat{D}$:

$$\hat{D} = \theta \left[\bar{Q} - \hat{Q}\right] = 0.2273.$$

  – Set $\hat{y} = 1$ and use (A.6) to compute $\hat{w}$:

$$\hat{w} = (1-\alpha)\hat{y} = 0.70.$$

  – Use (A.1)–(A.3) and ϕ_{mc} to obtain:

$$\hat{c}^y = \frac{\hat{w}}{1+\beta+\phi_{mc}} = 0.5330,$$

$$\hat{m} = \phi_{mc}\hat{c}^y = 0.2665\ 10^{-2},$$

$$\hat{c}^o = \beta(1+\hat{r})\hat{c}^y = 0.3447,$$

$$\hat{k} = \beta\hat{c}^y = 0.1643.$$

  – Use Ω in (A.7) to ensure that $\hat{y} = 1$:

$$\Omega = \hat{k}^{-\alpha} = 1.7190.$$

– Use δ to ensure that (A.5) holds:

$$\delta = \frac{\alpha}{\hat{k}} - \hat{r} = 0.7280.$$

– Use $\phi_{\gamma md}$ to compute γ:

$$\gamma = -\frac{1}{L\hat{m}} \ln\left(1 - \phi_{\gamma md}\right) = 7.5807\ 10^{-2}$$

– Use $\phi_{\gamma\eta}$ to compute η:

$$\eta = \frac{\gamma}{\phi_{\gamma\eta}} = 8.4230\ 10^{-2}$$

– Use χ to ensure that (A.4) holds:

$$\chi = \hat{m}\ \left[\frac{1}{\hat{c}^y} - \frac{\beta\gamma\zeta}{\hat{Q}}\right] = 4.8584\ 10^{-3}.$$

– Use ξ to ensure that (A.8) holds:

$$\xi = \frac{\hat{d}}{1 - \phi_{\gamma md}} = 2.3190\ 10^{-3}.$$

A.2 Fitting the non-linear environmental function

- We want an S-shaped fundamental difference equation for Q_t (featuring two stable steady states) as drawn in Figure 3(d).
- We choose six points through which the curve must pass. The first three are:
  – The unmanaged ME for the linear model provides the clean market equilibrium, $\hat{Q}_c^{me} = 2.5$ and $\hat{D}^{me} = 0.2273$.
  – In the DSO with a linear regeneration function we find $\hat{Q}^{so} = 2.7604$ and $\hat{D}^{so} = 0.1089$.
  – We also know the bliss point with $\hat{Q} = \bar{Q}$ and $\hat{D} = 0$.
- The last three points are:
  – We postulate a target value for the stable dirty equilibrium $\hat{Q}_d^{me} = 1$ and $\hat{D}^{me} = 0.2273$.
  – We postulate the disaster point $\hat{Q} = 0$ and $\hat{D} = D^{\max} = 1$.
  – We postulate an inflexion point at $\hat{Q}_{\inf} = \hat{Q}_d^{me} + \frac{1}{2}\left(\hat{Q}_c^{me} - \hat{Q}_d^{me}\right) = 1.7500$ and $\hat{D}^{me} = 0.2273$.

- We know (from the simulations) that $\hat{D}^{me}(Q)$ is virtually constant except for very low values of $\hat{Q}$.
- We ask Matlab:
  - To fit a fifth-degree polynomial through these points (using `polyfit`):

$$D = \Phi(Q) \equiv \phi_5 Q^5 + \phi_4 Q^4 + \phi_3 Q^3 + \phi_2 Q^2 + \phi_1 Q + \phi_0$$

  - The parameters are: $\phi_5 = 0.1121$, $\phi_4 = -0.9265$, $\phi_3 = 2.5646$, $\phi_2 = -2.4783$, $\phi_1 = 0.0447$, and $\phi_0 = 1.0000$.
  - The function is plotted in Figure 3(c).
- The environmental function outside the steady state is recovered as follows:
  - First we note that:

$$Q_{t+1} = H(Q_t) - D_t$$

    so that:

$$D_t = H(Q_t) - Q_{t+1}$$

  - Second we note that $\Phi(Q) = H(Q) - Q$ so that outside the steady state we obtain:

$$\begin{aligned} H(Q_t) &= Q_t + \Phi(Q_t) \\ &= \phi_5 Q_t^5 + \phi_4 Q_t^4 + \phi_3 Q_t^3 + \phi_2 Q_t^2 + (1 + \phi_1) Q_t + \phi_0. \end{aligned}$$

Appendix B: First-best social optimum in a deterministic world

The Lagrangian for the DSO is given by:

$$\begin{aligned}\mathcal{L}_t = &\sum_{\tau=0}^{\infty} \beta_G^{\tau-1} \left[\ln c_{t+\tau-1}^{y} + \chi \ln m_{t+\tau-1} + \beta \ln c_{t+\tau}^{o} + \zeta\beta \ln Q_{t+\tau}\right] \\ &+ \sum_{\tau=0}^{\infty} \beta_G^{\tau} \lambda_{t+\tau}^{k} \left[f(k_{t+\tau}) + (1-\delta)k_{t+\tau} - c_{t+\tau}^{y} - c_{t+\tau}^{o} - m_{t+\tau} - g_{t+\tau} - k_{t+\tau+1}\right] \\ &+ \sum_{\tau=0}^{\infty} \beta_G^{\tau} \lambda_{t+\tau}^{q} \left[H(Q_{t+\tau}) - \xi L f(k_{t+\tau}) e^{-\gamma L m_{t+\tau} + \eta L g_{t+\tau}} - Q_{t+\tau+1}\right],\end{aligned}$$

where $\lambda_{t+\tau}^{k}$ and $\lambda_{t+\tau}^{q}$ are the Lagrange multipliers. The first-order necessary conditions are (for $\tau = 0, 1, \ldots$):

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+\tau}^{y}} = \beta_G^{\tau}\left[\frac{1}{c_{t+\tau}^{y}} - \lambda_{t+\tau}^{k}\right] = 0, \tag{B.1}$$

$$\frac{\partial \mathcal{L}_t}{\partial m_{t+\tau}} = \beta_G^{\tau}\left[\frac{\chi}{m_{t+\tau}} - \lambda_{t+\tau}^{k} + \gamma L D_{t+\tau} \lambda_{t+\tau}^{d}\right] = 0, \tag{B.2}$$

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+\tau}^{o}} = \beta_G^{\tau}\left[\frac{\beta}{\beta_G c_{t+\tau}^{o}} - \lambda_{t+\tau}^{k}\right] = 0, \tag{B.3}$$

$$\frac{\partial \mathcal{L}_t}{\partial Q_{t+\tau+1}} = \beta_G^{\tau}\left[\frac{\beta\zeta}{Q_{t+\tau+1}} - \lambda_{t+\tau}^{q} + \beta_G \lambda_{t+\tau+1}^{q} H'(Q_{t+\tau+1})\right] = 0, \tag{B.4}$$

$$\begin{aligned}\frac{\partial \mathcal{L}_t}{\partial k_{t+\tau+1}} = \beta_G^{\tau}\Big[&- \lambda_{t+\tau}^{k} + \beta_G \left[f'(k_{t+\tau+1}) + 1 - \delta\right] \lambda_{t+\tau+1}^{k} \\ &- \beta_G \xi L f'(k_{t+\tau+1}) e^{-\gamma L m_{t+\tau+1} + \eta L g_{t+\tau+1}} \lambda_{t+\tau+1}^{q}\Big] = 0,\end{aligned} \tag{B.5}$$

$$\frac{\partial \mathcal{L}_t}{\partial g_{t+\tau}} = \beta_G^{\tau}\left[-\lambda_{t+\tau}^{k} + \eta L D_{t+\tau} \lambda_{t+\tau}^{q}\right] \leq 0, \quad g_{t+\tau} \geq 0, \quad g_{t+\tau}\frac{\partial \mathcal{L}_t}{\partial g_{t+\tau}} = 0, \tag{B.6}$$

$$D_{t+\tau} \equiv \xi L f(k_{t+\tau}) e^{-\gamma L m_{t+\tau} + \eta L g_{t+\tau}} \tag{B.7}$$

In principle this system, together with the equality constraints (37)–(39) and the initial conditions can be solved to find the solution to the most general version of the DSO.

In the main text we focus on the case for which $g_{t+\tau} > 0$ for all τ. This condition holds for the scenario illustrated in Figure 5. From (B.6) we find that $\lambda_{t+\tau}^{k} = \eta L D_{t+\tau} \lambda_{t+\tau}^{d}$ so that the set of first-order conditions simplifies to:

$$\frac{1}{c_{t+\tau}^{y}} = \frac{\beta}{\beta_G c_{t+\tau}^{o}} = \frac{\eta}{\eta-\gamma}\frac{\chi}{m_{t+\tau}} = \lambda_{t+\tau}^{k}, \tag{B.8}$$

$$\lambda_{t+\tau}^{q} = \frac{\beta\zeta}{Q_{t+\tau+1}} + \beta_G \lambda_{t+\tau+1}^{q} H'(Q_{t+\tau+1}), \tag{B.9}$$

$$\lambda_{t+\tau}^{k} = \beta_G\left[f'(k_{t+\tau+1}) + 1 - \delta - \frac{f'(k_{t+\tau+1})}{\eta L f(k_{t+\tau+1})}\right]\lambda_{t+\tau+1}^{k}. \tag{B.10}$$

Appendix C: Computing the stochastic models

In this appendix we provide some details on the computational methods used to simulate the stochastic models of the unmanaged market economy (SUME) and the first-best social optimum (SSO).

C.1 Unmanaged market equilibrium

In order to compute the market equilibrium we must find a convenient form for $\mathcal{M}(m_t, \tilde{Z}_t)$ which is defined in general as:

$$\mathcal{M}(m_t, k_t, Q_t) \equiv \mathbb{E}_t \left[\frac{\gamma \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t}}{H(Q_t) - \bar{\varepsilon} - \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t} + \varepsilon_{t+1}} \right]. \tag{C.1}$$

where $\bar{\varepsilon} = \theta \bar{Q}$ for the linear case and $\bar{\varepsilon} = \phi_0$ for the nonlinear case. For given values of m_t, k_t and Q_t we need to compute:

$$\mathbb{E}_t \left[\frac{\gamma \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t}}{H(Q_t) - \bar{\varepsilon} - \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t} + \varepsilon_{t+1}} \right] = \int_0^{\infty} \frac{\gamma \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t} \phi(x)}{H(Q_t) - \bar{\varepsilon} - \xi L \Omega k_t^{\alpha} e^{-\gamma L m_t} + x} dx, \tag{C.2}$$

where $\phi(x)$ is the probability density function of lognormal distribution:

$$\phi(x) \equiv \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right\}, \tag{C.3}$$

for which μ and σ are the location parameters. We require that:

$$E\left[\varepsilon_{t+1}\right] = \bar{\varepsilon}, \text{ and } V\left[\varepsilon_{t+1}\right] = \nu^2. \tag{C.4}$$

We set the standard deviation equal to $\nu = 0.2$ and obtain:

$$\mu = \ln \bar{\varepsilon} - \frac{1}{2} \ln \left[1 + \left(\frac{\nu}{\bar{\varepsilon}} \right)^2 \right], \quad \sigma = \sqrt{\ln \left[1 + \left(\frac{\nu}{\bar{\varepsilon}} \right)^2 \right]}. \tag{C.5}$$

Rather than computing the integral (C.2) with some quadrature-based routine we employ a Quasi-Monte-Carlo (QMC) method (Morokoff and Caflisch, 1995). In the first step we compute $N_h = 1000$ points of a Halton sequence in the interval $(0, 1)$ and denote these values as h_i (for $i = 1, \ldots, N_h$). Second, we note that for each h_i we can find a point x_i on the cumulative density function (CDF) of the lognormal, i.e.:

$$h_i = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left[\frac{\ln x_i - \mu}{\sigma\sqrt{2}} \right], \tag{C.6}$$

which can be solved for x_i:

$$x_i = \exp\left\{ \mu + \sigma\sqrt{2} \operatorname{erf}^{-1}(2h_i - 1) \right\}, \quad (\text{for } i = 1, \ldots, N_h), \tag{C.7}$$

where $\text{erf}^{-1}(x)$ is the inverse error function. Third, we approximate the integral in (C.2) by computing:

$$\mathbb{E}_t\left[\frac{\gamma\xi L\Omega k_t^{\alpha}e^{-\gamma L m_t}}{H(Q_t)-\bar{\varepsilon}-\xi L\Omega k_t^{\alpha}e^{-\gamma L m_t}+\varepsilon_{t+1}}\right]\approx\frac{1}{N_h}\sum_{i=1}^{N_h}\frac{\gamma\xi L\Omega k_t^{\alpha}e^{-\gamma L m_t}}{H(Q_t)-\bar{\varepsilon}-\xi L\Omega k_t^{\alpha}e^{-\gamma L m_t}+x_i}. \quad \text{(C.8)}$$

Using this approach it is straightforward to find the policy function for private abatement, $\mathbf{m}(k_t, Q_t)$, by solving equation (50) numerically.

For the unmanaged market economy we simulate the follow system of stochastic difference equations:

$$k_{t+1}=\frac{\beta}{1+\beta}\left[\Omega k_t^{\alpha}-\mathbf{m}(k_t,Q_t)\right], \quad \text{(C.9)}$$

$$Q_{t+1}=H(Q_t)-\bar{\varepsilon}-\xi L\Omega k_t^{\alpha}e^{-\gamma L\mathbf{m}(k_t,Q_t)}+\varepsilon_{t+1}, \quad \text{(C.10)}$$

where ε_{t+1} is drawn from a lognormal distribution with mean $\bar{\varepsilon}$ and standard deviation ν. In the simulations we enforce a strict upper bound on environmental quality, i.e. $Q_t < \bar{Q}$ at all times (since 'better than paradise' is impossible). Furthermore we start the simulations using the deterministic steady-state values for k_t and Q_t as initial conditions. In the text we illustrate the nonlinear case, with $H(Q_t)$ as given in (35). Figure C.1 illustrates the dynamics of the stochastic model when the environmental regeneration function is linear (as in (33)). In producing these figures we adopt a fixed seed and thus use the same quasi-random vector of standard-normal variables $z_{t+1} \sim N(0,1)$ in order to generate the vector of ε_{t+1} values such that $\varepsilon_{t+1} = e^{\mu+\sigma z_{t+1}}$.

C.2 First-best social optimum

The Bellman equation is stated in equation (62) in the text and restated here:

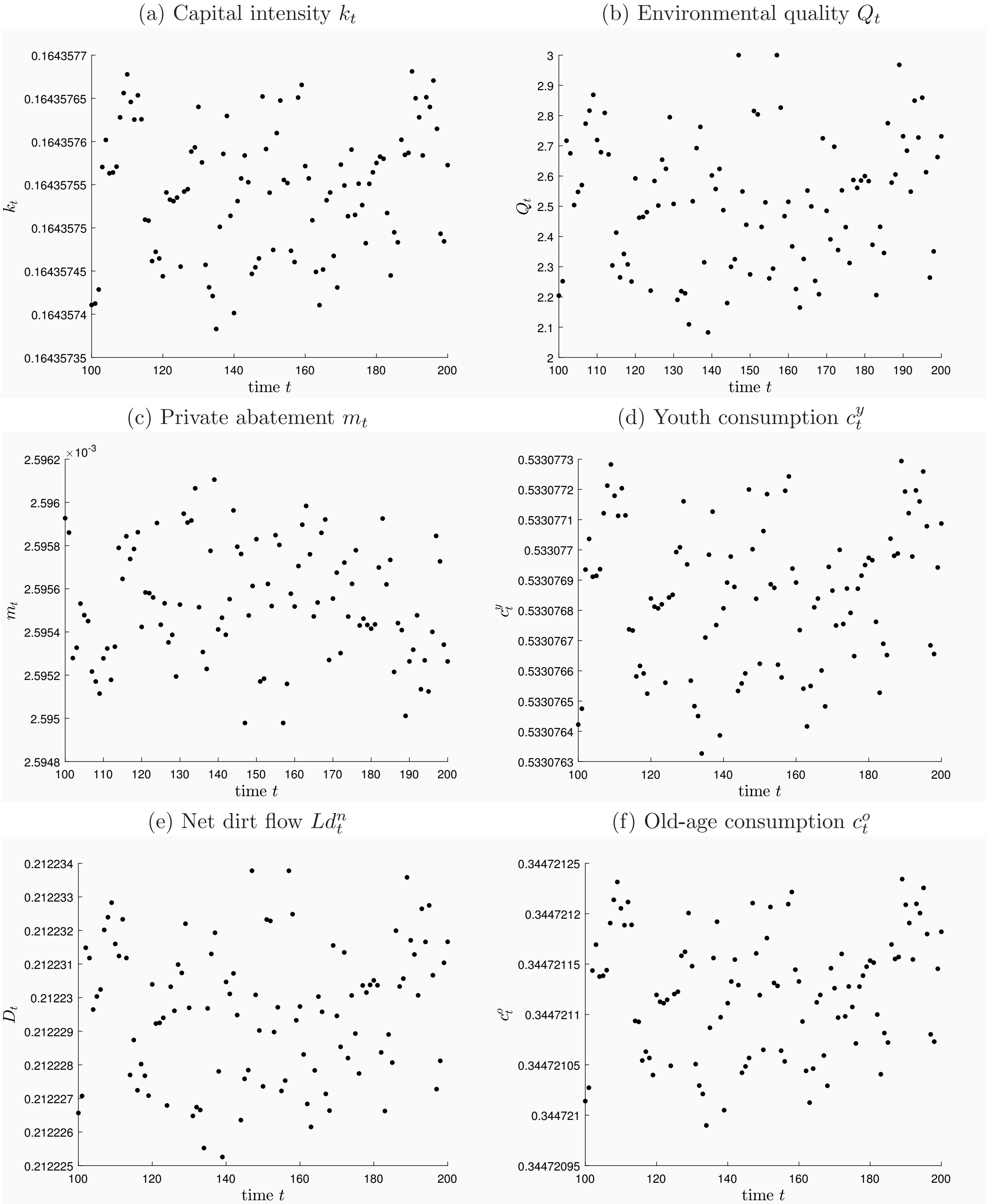
$$\mathcal{V}(k_t,Q_t)=\max_{\{c_t^y,m_t,c_t^o,g_t\}} SF(c_t^y,m_t,c_t^o,Q_t)+\beta_G\mathbb{E}_t\left[\mathcal{V}(k_{t+1},Q_{t+1})\right] \quad \text{(C.11)}$$

$$\begin{aligned}\text{s.t.}\quad & k_{t+1}=f(k_t)+(1-\delta)k_t-c_t^y-c_t^o-m_t-g_t\\ & \hat{Q}_{t+1}=H(Q_t)-\bar{\varepsilon}-D_t,\quad Q_{t+1}=\hat{Q}_{t+1}+\varepsilon_{t+1}\\ & D_t\ \equiv\xi Lf(k_t)e^{-\gamma Lm_t-\eta Lg_t},\\ & g_t\geq 0.\end{aligned} \quad \text{(C.12)}$$

The Lagrangian for the optimization problem on the right-hand side of (C.11) is given by:

$$\begin{aligned}\mathcal{L}\equiv{}&\ln c_t^y+\chi\ln m_t+\frac{\beta}{\beta_G}\left[\ln c_t^0+\zeta\ln Q_t\right]\\ &+\beta_G\mathbb{E}_t\left[\mathcal{V}(f(k_t)+(1-\delta)k_t-c_t^y-c_t^o-m_t-g_t,H(Q_t)-\bar{\varepsilon}-D_t+\varepsilon_{t+1})\right]\\ &+\lambda_t^d\left[D_t-\xi Lf(k_t)e^{-\gamma Lm_t-\eta Lg_t}\right],\end{aligned}$$

Figure C.1: The unmanaged economy in a stochastic world



and the first-order necessary conditions for c_t^y, c_t^o, m_t, g_t, and D_t are:

$$\frac{1}{c_t^y} = \beta_G \mathbb{E}_t \left[\frac{\partial \mathcal{V}\left(k_{t+1}, \hat{Q}_{t+1} + \varepsilon_{t+1}\right)}{\partial k_{t+1}} \right], \tag{C.13}$$

$$\frac{\beta}{\beta_G c_t^o} = \beta_G \mathbb{E}_t \left[\frac{\partial \mathcal{V}\left(k_{t+1}, \hat{Q}_{t+1} + \varepsilon_{t+1}\right)}{\partial k_{t+1}} \right], \tag{C.14}$$

$$\frac{\chi}{m_t} = \beta_G \mathbb{E}_t \left[\sum_{i=1}^{N_h} \frac{\partial \mathcal{V}\left(k_{t+1}, \hat{Q}_{t+1} + \varepsilon_{t+1}\right)}{\partial k_{t+1}} \right] - \gamma L D_t \lambda_t^d, \tag{C.15}$$

$$\frac{\partial \mathcal{L}}{\partial g_t} \equiv -\beta_G \mathbb{E}_t \left[\frac{\partial \mathcal{V}\left(k_{t+1}, \hat{Q}_{t+1} + \varepsilon_{t+1}\right)}{\partial k_{t+1}} \right] + \eta L D_t \lambda_t^d \leq 0, \quad g_t \geq 0, \quad g_t \frac{\partial \mathcal{L}}{\partial g_t} = 0, \tag{C.16}$$

$$\frac{\partial \mathcal{L}}{\partial D_t} \equiv -\beta_G \mathbb{E}_t \left[\frac{\partial \mathcal{V}\left(k_{t+1}, \hat{Q}_{t+1} + \varepsilon_{t+1}\right)}{\partial Q_{t+1}} \right] + \lambda_t^d = 0. \tag{C.17}$$

We define:

$$\lambda_t^k \equiv \beta_G \mathbb{E}_t \left[\frac{\partial \mathcal{V}\left(k_{t+1}, \hat{Q}_{t+1} + \varepsilon_{t+1}\right)}{\partial k_{t+1}} \right], \tag{C.18}$$

$$\lambda_t^q \equiv \beta_G \mathbb{E}_t \left[\frac{\partial \mathcal{V}\left(k_{t+1}, \hat{Q}_{t+1} + \varepsilon_{t+1}\right)}{\partial Q_{t+1}} \right], \tag{C.19}$$

and derive from (C.13)–(C.15) and (C.17):

$$\lambda_t^k = \frac{1}{c_t^y} = \frac{\beta}{\beta_G c_t^o} = \frac{\chi}{m_t} + \gamma L D_t \lambda_t^q. \tag{C.20}$$

Differentiating the left-hand side of the value function (C.11) with respect to k_t and Q_t gives:

$$\begin{aligned}
\frac{\partial \mathcal{V}(k_t, Q_t)}{\partial k_t} &= (f'(k_t) + 1 - \delta)\lambda_t^k - \xi L f'(k_t) e^{-\gamma L m_t - \eta L g_t} \lambda_t^q, \\
\frac{\partial \mathcal{V}(k_t, Q_t)}{\partial Q_t} &= \frac{\beta \zeta}{\beta_G} \frac{1}{Q_t} + H'(Q_t)\lambda_t^q.
\end{aligned}$$

It follows that:

$$\begin{aligned}
\frac{\partial \mathcal{V}(k_{t+1}, Q_{t+1})}{\partial k_{t+1}} &= (f'(k_{t+1}) + 1 - \delta)\lambda_{t+1}^k - \xi L f'(k_{t+1}) \lambda_{t+1}^d, \\
\frac{\partial \mathcal{V}(k_{t+1}, Q_{t+1})}{\partial Q_{t+1}} &= \frac{\beta \zeta}{\beta_G} \frac{1}{\hat{Q}_{t+1} + \varepsilon_{t+1}} + H'(\hat{Q}_{t+1} + \varepsilon_{t+1})\lambda_{t+1}^q,
\end{aligned}$$

and (by using (C.18) and (C.19)):

$$\lambda_t^k = \beta_G \mathbb{E}_t\Big[(f'(k_{t+1}) + 1 - \delta)\lambda_{t+1}^k - \xi f'(k_{t+1})e^{-\gamma L m_{t+1} - \eta L g_{t+1}}\lambda_{t+1}^q\Big], \tag{C.21}$$

$$\lambda_t^q = \mathbb{E}_t\left[\frac{\beta\zeta}{\hat{Q}_{t+1} + \varepsilon_{t+1}} + \beta_G H'(\hat{Q}_{t+1} + \varepsilon_{t+1})\lambda_{t+1}^q\right]. \tag{C.22}$$

The SSO is characterized by equations (C.20)–(C.22), the evolution equations:

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t^y - c_t^o - m_t - g_t, \tag{C.23}$$

$$Q_{t+1} = \hat{Q}_{t+1} + \varepsilon_{t+1}, \tag{C.24}$$

$$\hat{Q}_{t+1} = H(Q_t) - \bar{\varepsilon} - D_t, \tag{C.25}$$

$$D_t \equiv \xi L f(k_t) e^{-\gamma L m_t - \eta L g_t}, \tag{C.26}$$

and the complementarity condition:

$$A_t \equiv \lambda_t^k - \eta L D_t \lambda_t^q, \tag{C.27}$$

$$0 = A_t + g_t - \sqrt{A_t^2 + g_t^2}, \tag{C.28}$$

where (C.28) employs the Fischer-Burmeister complementarity function (satisfying $\phi(a,b) = 0 \Leftrightarrow a \geq 0,\ b \geq 0$, and $ab = 0$).