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Tapan Mitra
Geir B. Asheim
Wolfgang Buchholz
Cees Withagen

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Abstract

The Dasgupta-Heal-Solow-Stiglitz model of capital accumulation and resource depletion poses the following sustainability problem: is it feasible to sustain indefinitely a level of consumption that is bounded away from zero? We provide a complete technological characterization of the sustainability problem in this model without reference to the time path. As a byproduct we show general existence of a maximin optimal path under weaker conditions that those employed in previous work. Our proofs yield new insights into the meaning and significance of Hartwick's reinvestment rule.

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Keywords: sustainability, maximin.

Tapan Mitra
Department of Economics
Cornell University
418 Uris Hall
USA – Ithaca, New York 14853
tm19@cornell.edu

Geir B. Asheim
Department of Economics
University of Oslo
P.O. Box 1095 Blindern
Norway – 0317 Oslo
g.b.asheim@econ.uio.no

Wolfgang Buchholz
Department of Economics
University of Regensburg
Germany – 93040 Regensburg
wolfgang.buchholz@wiwi.uni-regensburg.de

Cees Withagen
Department of Economics
VU University of Amsterdam
De Boelelaan 1105
The Netherlands – 1081 HV Amsterdam
cwithagen@feweb.vu.nl

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1 Introduction

Models of capital accumulation and resource depletion have been analyzed since the Dasgupta-Heal-Solow-Stiglitz (DHSS) (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974) model was introduced in the early 1970s. They allow the analyst to pose the question of whether and how accumulation in augmentable capital can make up for the depletion of an exhaustible and non-renewable resource. Their relevance is not only tied to the limited supply of natural resources (like oil and gas) as, in the very long run, the atmosphere's cumulative capacity for absorbing CO_2 without causing serious climate change is a non-renewable and exhaustible resource. Thus, such models can be used to enhance our understanding of intergenerational conflicts.

The DHSS model poses three interesting and separate problems.

(1) *The sustainability problem:* is it feasible to sustain indefinitely a level of consumption that is bounded away from zero?

(2) *The maximin existence problem:*[1] is there a maximum level of consumption that can be sustained?

(3) *The maximin efficiency problem:* is keeping consumption equal to the maximum sustainable level efficient?

Whether there is a non-trivial sustainable path in the DHSS model matters also for the criteria of undiscounted utilitarianism (Dasgupta and Heal, 1979, sect. 10.3), sustainable discounted utilitarianism (Asheim and Mitra, 2010, sect. 5), and (extended) rank-discounted utilitarianism (Zuber and Asheim, 2011, sect. 6.2): in the former case no optimal path exists while in the latter cases all paths are equally bad if positive consumption cannot be sustained. Hence, the sustainability problem has relevance also if one does not ascribe to maximin as an extreme egalitarian criterion. Posing these questions thus responds to a call from Koopmans (1965, p. 228–229):

> "... to argue against the complete separation of the ethical or political choice of an objective function from the investigation of the set of technologically feasible paths. ... Ignoring realities in adopting 'principles' may lead one to search for a nonexistent optimum, or to adopt an 'optimum' that is open to unanticipated objections."

[1]Note that even when the set of positive consumption levels that can be sustained indefinitely is non-empty and bounded, it need not contain its least upper bound. We provide an example of this at the end of Section 3.

Solow (1974) was first to pose these problems. He applied the maximin criterion in the context of the DHSS model, whereby the consumption of the worst-off generation is maximized. By letting output be a Cobb-Douglas function of capital and resource input, Solow (1974) solved the maximin problem in the continuous time DHSS model by calculating closed-form solutions. He showed that an efficient and egalitarian maximin path with positive consumption exists if and only if the constant elasticity of output with respect to augmentable capital, a, exceeds that with respect to resource input, b. Moreover, if this condition is *not* satisfied, the greatest lower bound for consumption is zero, so that no positive level of consumption can be sustained indefinitely and any path solves the maximin problem. Together, these observations show that the solution of the sustainability problem depends on whether $a > b$ and establish general maximin existence in the Cobb-Douglas case.

The condition $a > b$ is a requirement on the production function in the special Cobb-Douglas class. Is it possible to generalize such a condition on the production function to a much more general class? Does there exist a complete technological characterization of the sustainability problem that depends only on the production function and the initial stocks and does not involve any reference to time paths? In Theorems 1 and 2 we establish that this is indeed possible under weak assumptions on the production function (other than usual neoclassical properties, the property that positive output requires positive inputs of both capital and resource). We do so by providing a condition which takes the form of an integral criterion that can be checked directly using information only about the production function.

The constructive strategy on which our characterization is based yields new important insights into the meaning and significance of Hartwick's rule for reinvesting resource rents. Hartwick's rule prescribes that the value of resource depletion be reinvested in augmentable capital. As is well-known (Hartwick, 1977; Dixit, Hammond and Hoel, 1980), following Hartwick's rule leads to constant consumption if technology is stationary. However, as suggested already by Buchholz (1984, pp. 69–70) (see also Martinet and Doyen, 2007, p. 25), following Hartwick's rule also maximizes the ratio of capital accumulation to resource depletion for a given level of consumption. Thus, in the capital stock/resource stock space, Hartwick's rule can be understood as a prescription for maximal conservation of the economy's productive capacity, subject to the constraint imposed by maintaining current consumption.

Our approach is related to the Cass-Mitra (Cass and Mitra, 1991) integral criterion characterization of the sustainability problem, which also translates information about time paths to information about the technology. However, the present characterization uses a resource requirement function along a path which satisfies Hartwick's rule, and this is absent (at least directly) in the Cass-Mitra characterization, which uses the resource requirement function along isoquants of the production function. That is, the present characterization focuses directly on maintaining a constant consumption (by following Hartwick's rule), while the Cass-Mitra characterization focuses on behavior associated with maintaining constant output as a means to providing a consumption stream that is bounded away from zero.[2]

By showing that the set of consumption levels for which a path satisfying Hartwick's rule exists, if non-empty, is bounded and contains its least upper bound, we can use the techniques developed for Theorems 1 and 2 to show that there is always a maximum level of consumption that can be sustained, as reported in Theorem 3. Thus, we establish existence of a maximin path in the continuous-time DHSS model under weaker assumptions than those imposed in previous literature. In particular, we consider a technological environment where (contrary to the Cobb-Douglas case) we cannot prove that an egalitarian path with consumption equal to the maximin level is efficient even if a path with consumption bounded away from zero exists.

The efficient and egalitarian maximin path in the Cobb-Douglas version of the DHSS model is an example of a *regular maximin path* (Burmeister and Hammond, 1977; Dixit, Hammond and Hoel, 1980), being a price supported constant consumption path with finite present value of future consumption and which satisfies a capital value transversality condition and resource exhaustion. Regularity entails that reduced consumption on a finite time interval can be transformed into a uniform addition to consumption for the rest of the path and implies that any maximin path is egalitarian and efficient. In the discrete-time setting, conditions for regularity were studied by Dasgupta and Mitra (1983) and Cass and Mitra (1991), while in the continuous-time literature on maximin paths, it has been common simply to assume regularity (see, e.g., Withagen and Asheim, 1998; Cairns and Long, 2006). In Theorem 4 we show that under an additional assumption (namely that the production function has the property that the resource is 'important') the egalitarian path

[2]The Cass-Mitra characterization is in a discrete-time model, where (as noted by Dasgupta and Mitra, 1983) Hartwick's rule does not hold for efficient equitable paths (which are always maximin paths as well). So, following Hartwick's rule was not a natural benchmark in that setting. The Cass-Mitra characterization also uses a set of minimal standard assumptions on the technology; in particular, smoothness conditions are not used there.

following Hartwick's rule with consumption equal to the maximin level is efficient if there exists a path with consumption bounded away from zero.

The paper is organized as follows. We introduce the DHSS model with its assumptions in Section 2. Then in Section 3 we provide an intuitive explanation of our strategy of proof and how it yields insights into the meaning and significance of Hartwick's rule, before presenting our complete technological characterization of the sustainability problem and our maximin existence result. The formal proofs of these theorems are presented Sections 4-7. We finally discuss the issue of maximin efficiency in Section 8 and offer concluding remarks in Section 9.

2 Preliminaries

Denote by k the stock of an augmentable capital good (which is assumed to be non-depreciating) and by r the flow of an exhaustible resource input. Denote by $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ the production function for the capital/consumption good, employing k and r as inputs. The output $F(k, r)$ is used to provide a flow of consumption, c, or to augment the capital stock through a flow of net investment, $\dot{k}$. Output $F(k, r)$ is the only source of consumption or net investment.

Throughout we will impose the following three assumptions on F (where subscript 1 signifies the partial derivative w.r.t. the first variable, etc.):

Assumption 1 (A1) $F(0, r) = F(k, 0) = 0$ for $k \geq 0$ and $r \geq 0$.

Assumption 2 (A2) F is continuous, concave and nondecreasing on $\mathbb{R}^2_+$.

Assumption 3 (A3) F is twice continuously differentiable on $\mathbb{R}^2_{++}$, with $F_1(k, r) > 0$, $F_2(k, r) > 0$, and $F_{22}(k, r) < 0$ for $(k, r) \in \mathbb{R}^2_{++}$.

Let $(k_0, m_0) \gg 0$ be a vector of initial stocks of capital and resource. A *path* from (k_0, m_0) is a triplet of functions $(c(t), k(t), r(t))$, with $c(\cdot) : [0, \infty) \to \mathbb{R}_+$, $k(\cdot) : [0, \infty) \to \mathbb{R}_+$ and $r(\cdot) : [0, \infty) \to \mathbb{R}_+$, where $k(t)$ is differentiable and $(c(t), r(t))$ are continuous, and where

$$c(t) = F(k(t), r(t)) - \dot{k}(t)\,, \tag{1}$$

$$k(0) = k_0\,, \tag{2}$$

$$\int_0^\infty r(t)dt \leq m_0\,. \tag{3}$$

Write $m(\cdot) : [0, \infty) \to \mathbb{R}_+$ for the associated function of remaining resource stock:

$$m(t) = m_0 - \int_0^t r(\tau) d\tau \quad \text{for } t \geq 0 \,.$$

Note that along any path $(c(t), k(t), r(t))$ from (k_0, m_0), both $k(t)$ and $m(t)$ are continuously differentiable functions of t.

A triple $(c, k, r) \gg 0$ satisfies *Hartwick's* reinvestment *rule* (Hartwick, 1977; Dixit, Hammond and Hoel, 1980) if

$$F(k, r) - c = F_2(k, r) r \,. \tag{HaR}$$

A path $(c(t), k(t), r(t))$ from (k_0, m_0) is *interior* if $k(t) > 0$ and $r(t) > 0$ for all $t \geq 0$. A path $(c(t), k(t), r(t))$ from (k_0, m_0) is *egalitarian* if there is $c \geq 0$ such that $c(t) = c$ for all $t \geq 0$. A path $(c(t), k(t), r(t))$ from (k_0, m_0) is a *maximin path* if

$$\inf_{t \geq 0} c(t) \geq \inf_{t \geq 0} c'(t) \tag{2}$$

for every path $(c'(t), k'(t), r'(t))$ from (k_0, m_0). A maximin path is *non-trivial* if, in addition, $\inf_{t \geq 0} c(t) > 0$.

Assumptions **A1**–**A3** do not imply that it is feasible to sustain a path from (k_0, m_0) with consumption bounded away from zero. Therefore, the set

$$C(k_0, m_0) \equiv \{c \in \mathbb{R}_{++} : \exists (c(t), k(t), r(t)) \text{ from } (k_0, m_0) \text{ with } c(t) \geq c \text{ for } t \geq 0\}$$

of positive sustainable consumption levels need not be non-empty. In particular, if

$$F(k, r) = k^a r^b \text{ for } (k, r) \in \mathbb{R}^2_+, \text{ with } a > 0,\ b > 0 \text{ and } a + b \leq 1, \tag{4}$$

then assumptions **A1**–**A3** are clearly satisfied. However, as shown by Solow (1974, sect. 8 & App. B),[3] the set $C(k_0, m_0)$ is non-empty if and only if $a > b$.

Solow (1974, p. 34) provided a justification for restricting his analysis to the Cobb-Douglas case, where his footnote 2 is particularly persuasive:

> "Only the Cobb-Douglas will do among CES functions. If the elasticity of substitution between resources and other factors exceeds one, then resources are not indispensable to production. If it is less than one, then the average product of resources is bounded. So only the Cobb-Douglas remains."

[3] Solow (1974) did not provide an analysis of the borderline case in which $a = b$.

However, several authors subsequently investigated the sustainability problem under more general production functions. E.g., Dasgupta and Heal (1979, p. 226) showed that if the elasticity of substitution σ between k and r depends on the ratio r/k only, then what counts is $\liminf \sigma(r/k)$ when r/k goes to zero: $C(k_0, m_0)$ is empty if it is smaller than 1, while $C(k_0, m_0)$ is non-empty if it is greater than 1 or if it is equal to 1 and the share of capital exceeds that of the resource.

Others felt that instead of separating the issues of (i) comparing a and b in the Cobb-Douglas case ($\sigma = 1$), and (ii) whether $\sigma \lesseqgtr 1$ for the class of CES functions, Solow's (and Dasgupta and Heal's related) observations on these two issues should follow from a more general unifying principle. This led naturally to an investigation of how the nature of the isoquant map (between k and r) characterizes sustainability. Almost simultaneously, and independently, this approach was developed by Mitra (1978b), Buchholz (1982), and Shimomura (1983). For an account of this line of enquiry, see Kemp, Long and Shimomura (1984).

The isoquant map is easy to study in the case in which the production function $F(k, r)$ is homogeneous of degree one in (k, r), as assumed by Dasgupta and Heal (1974), but not by Solow (1974) (and not here). For, in this case, each isoquant is simply a scaled version of the unit output isoquant. The criterion for sustainability then turns out to be:

$$\int_{k_0}^{\infty} \tilde{\mathbf{r}}(k) dk < \infty \, ,$$

where $\tilde{\mathbf{r}}(k)$ is defined by $F(k, \tilde{\mathbf{r}}(k)) = 1$. That is, the "area under the unit isoquant" from k_0 to infinity be finite. Given this criterion, which was established in discrete time by Mitra (1978b) and in continuous time by Buchholz (1982), one can easily derive the observations of Solow (1974) regarding the roles of the (i) a and b in the Cobb-Douglas case, and (ii) σ for the class of CES functions, as special cases.

A more general criterion, not relying on smoothness assumptions on the production function, and the assumption that $F(k, r)$ be homogeneous of degree one, was established by Cass and Mitra (1979), which was eventually published as Cass and Mitra (1991). Few papers have subsequently been concerned with the sustainability problem, Martinet and Doyen (2007) being one exception. The present paper adds to the literature by using Hartwick's rule to yield an easily interpretable characterization of the sustainability problem, with a strategy of proof that as a by-product also yields results on maximin existence and efficiency.

3 Hartwick's rule as a resource requirement function

We first state a lemma which is the foundation for our main results. Let

$$D = \{(c,k) \in \mathbb{R}^2_{++} : \text{there exists } r > 0 \text{ such that } F(k,r) > c\}$$

be a domain set of capital-consumption pairs which allow for positive capital accumulation, and furthermore, define the associated correspondence:

$$D(k) = \{c \in \mathbb{R}_{++} : (c,k) \in D\} \ \text{ for } k \in \mathbb{R}_{++} \,.$$

Note that for every $k > 0$, $D(k)$ is non-empty, since every $0 < c < F(k,1)$ belongs to $D(k)$. In fact, it is easy to verify that $D(k)$ is the interval $(0, \lim_{r\to\infty} F(k,r))$.

We establish a *resource requirement function* defined on the domain set, D.

Lemma 1 *Assume that F satisfies **A1**–**A3**, and let $(c,k) \in D$. Then, there is a unique solution $\mathbf{r}(c,k)$ to*

$$\max_{r>0} \frac{F(k,r) - c}{r} \,. \tag{5}$$

Furthermore, (c,k,r) satisfies (HaR) if and only if $r = \mathbf{r}(c,k)$. Finally, the function $\mathbf{r} : D \to \mathbb{R}_{++}$ is continuously differentiable with

$$\mathbf{r}_1(c,k) > 0 \quad \textit{for all } (c,k) \in D \,. \tag{6}$$

It follows from Lemma 1, which will be proven in Section 4, that Hartwick's rule (cf. (HaR))—prescribing that the value of resource depletion be reinvested in augmentable capital—is satisfied if and only if $r = \mathbf{r}(c,k)$:

$$F(k, \mathbf{r}(c,k)) - c = F_2(k, \mathbf{r}(c,k))\mathbf{r}(c,k) \,.$$

However, following (Buchholz, 1984, pp. 69–70),[4] Lemma 1 shows that $\mathbf{r}(c,k)$ has another interpretation: Since $\dot{k} = F(k,r) - c$ and $\dot{m} = -r$, it is the rate of resource depletion that maximizes the ratio of capital accumulation to resource depletion,

$$\frac{\dot{k}}{-\dot{m}} = \frac{F(k,r) - c}{r} \,,$$

subject to the rate of consumption being equal to c. In other words,

$$\frac{\mathbf{r}(c,k)}{F(k, \mathbf{r}(c,k)) - c} = \frac{1}{F_2(k, \mathbf{r}(c,k))}$$

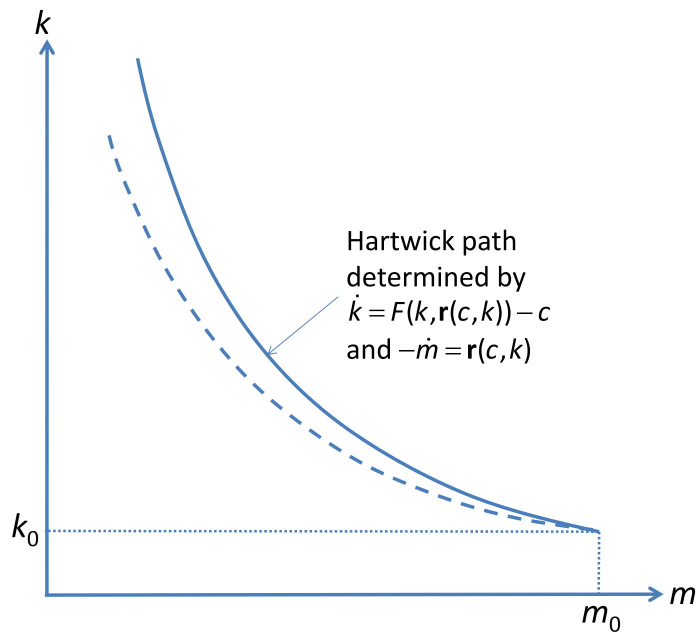

Figure 1: Maximizing the capital accumulation/resource depletion ratio

is the required resource depletion per unit capital accumulation if consumption is to be maintained at c. This observation is crucial for the main results of this paper.

To sketch the arguments on which our proofs are based, let $c \in D(k_0)$ and consider the path in (k, m) space determined by $(k(0), m(0)) = (k_0, m_0) \gg 0$ and, for all $t \geq 0$, $\dot{k}(t) = F(k(t), \mathbf{r}(c, k(t))) - c$ and $-\dot{m}(t) = \mathbf{r}(c, k(t))$, as illustrated by the solid line path in Figure 1. Refer to this path, provided that it exists, as the *Hartwick path* from (k_0, m_0) given the consumption level c.

If the accumulated resource input along the Hartwick path from (k_0, m_0) given the consumption level c does not exceed the initial resource stock m_0, then it is clearly feasible to sustain indefinitely a flow of consumption that is bounded away from zero. Since the accumulated resource input can be found by integrating the required resource depletion per unit capital accumulation from k_0 to ∞, without reference to the time path, this leads to the following sufficient condition for $C(k_0, m_0)$ being non-empty:

[4] In their Prop. 3 and subsequent discussion, Martinet and Doyen (2007) provide similar insights in the Cobb-Douglas case, using the viable control approach.

Theorem 1 *Assume that F satisfies **A1**–**A3**, and let $(k_0, m_0) >> 0$ be given. If*

$$\inf_{c \in D(k_0)} \int_{k_0}^{\infty} \frac{1}{F_2(x, \mathbf{r}(c, x))} dx < m_0 \,, \tag{7}$$

then $C(k_0, m_0)$ is non-empty.

Among all paths from (k_0, m_0), the Hartwick path maximizes at each point in (k, m) space the ratio of capital accumulation to resource depletion subject to the rate of consumption being equal to c. This reasoning indicates that any path with consumption bounded below by c deviating from the Hartwick path in (k, m) space must follow a trajectory like the one illustrated by the dashed line path in Figure 1.

To sustain consumption weakly above c indefinitely, the accumulated stock of augmentable capital must substitute for the flow of resource input so that the initial resource stock, m_0, is never exhausted. In Figure 1 this means that the path in (k, m) space must never intersect the vertical axis. Hence, if it is feasible to sustain consumption weakly above c, then there exists a Hartwick path from (k_0, m_0) given c, as there is no path to the north-east of the Hartwick path that sustains consumption weakly above c. By integrating the required resource depletion per unit capital accumulation from k_0 to ∞, without reference to the time path, this leads to the following necessary condition for $C(k_0, m_0)$ being non-empty:

Theorem 2 *Assume that F satisfies **A1**–**A3**, and let $(k_0, m_0) >> 0$ be given. If $C(k_0, m_0)$ is non-empty, then*

$$\inf_{c \in D(k_0)} \int_{k_0}^{\infty} \frac{1}{F_2(x, \mathbf{r}(c, x))} dx < m_0 \,. \tag{7}$$

The proofs of Theorems 1 and 2 show that if $C(k_0, m_0)$ is non-empty and c is in the interior of $C(k_0, m_0)$, then there is a Hartwick path from (k_0, m_0) given c. Furthermore, it holds that the set of consumptions levels c for which there is a Hartwick path from (k_0, m_0) given c is bounded above and contains its least upper bound. It therefore follows that $C(k_0, m_0)$ is bounded above and contains its least upper bound $\bar{c}$, and sustaining consumption at $\bar{c}$ can be implemented by following the Hartwick path from (k_0, m_0) given $\bar{c}$. This establishes existence of a maximin path from (k_0, m_0) if $C(k_0, m_0)$ is non-empty. If, on the other hand, $C(k_0, m_0)$ is empty and $(c(t), k(t), r(t))$ is a path from (k_0, m_0), then $\inf_{t \geq 0} c(t) = 0$. Hence, all paths are trivially maximin. Moreover, $(c^0(t), k^0(t), r^0(t))$ with $c^0(t) = 0$, $k^0(t) = k_0$ and $r^0(t) = 0$ is a path from (k_0, m_0), showing that the set of maximin paths is non-empty also in this case. This establishes maximin existence:

Theorem 3 *Assume that F satisfies **A1**–**A3**, and let $(k_0, m_0) >> 0$ be given. There exists a maximin path from (k_0, m_0).*

To turn these sketches into stringent proofs, various issues must be addressed. We do so in the subsequent Sections 4–7. Throughout these proofs, we assume that F satisfies **A1**–**A3**. Before doing so, we illustrate our results by examples.

We first illustrate Theorems 1 and 2 by showing how condition (7)—the criterion for $C(k_0, m_0)$ to be non-empty—corresponds to $a > b$ in the Cobb-Douglas version of the DHSS model considered by Solow (1974), where the production function is given by (4). As already noted, assumptions **A1**–**A3** are clearly satisfied.

It is easy to check that $D = \mathbb{R}^2_{++}$ and $D(k) = (0, \infty)$ for each $k \in \mathbb{R}_{++}$. Thus, $\mathbf{r}$ is a function from $\mathbb{R}^2_{++}$ to $\mathbb{R}_{++}$. Given any $(c, k) \in \mathbb{R}^2_{++}$, $\mathbf{r}(c, k)$ satisfies (HaR), and it is straightforward to verify that:

$$\mathbf{r}(c, k) = \frac{c^{(1/b)}}{(1-b)^{(1/b)} k^{(a/b)}} \quad \text{for all } (k, c) \in \mathbb{R}^2_{++}\,. \tag{8}$$

Then the criterion (7) for $C(k_0, m_0)$ to be non-empty translates to

$$\inf_{c>0} \left[\frac{c^{(1-b)/b}}{b(1-b)^{(1-b)/b}} \right] \int_{k_0}^{\infty} \frac{1}{x^{(a/b)}} dx < m_0 \tag{9}$$

by using (8). Clearly, with $(k_0, m_0) \gg 0$, (9) holds if and only if $a > b$, which is the result of Solow (1974, sect. 8 & App. B).

Following Dasgupta and Heal (1979, Sect. 7.2) by considering the class of CES production functions beyond the Cobb-Douglas case, so that

$$F(k, r) = \left(a k^{\frac{\sigma-1}{\sigma}} + b r^{\frac{\sigma-1}{\sigma}} + (1 - a - b) \right)^{\frac{\sigma}{\sigma-1}}$$

for $(k, r) \in \mathbb{R}^2_+$, with $a > 0$, $b > 0$, $a + b \leq 1$ and $\sigma > 0$, $\sigma \neq 1$, we have that:

- If $\sigma < 1$, then assumptions **A1**–**A3** are satisfied, but (7) cannot hold since

  $$\frac{1}{F_2(k, \mathbf{r}(c, k))} = \frac{\mathbf{r}(c, k)}{F(k, \mathbf{r}(c, k)) - c} \geq \frac{\mathbf{r}(c, k)}{F(k, \mathbf{r}(c, k))} \geq b^{\frac{\sigma-1}{\sigma}}$$

  if $c \in D(k_0)$. This confirms the well-known result that $C(k_0, m_0)$ is empty.

- If $\sigma > 1$, then assumption **A1** is not satisfied—so that Theorems 1 and 2 do not apply—while clearly $C(k_0, m_0)$ is non-empty.

Finally, by going to the limiting case where inputs are perfect substitutes ($\sigma = \infty$), we can provide an example where $C(k_0, m_0)$ is non-empty and bounded, but does not contain its least upper bound, thereby showing the significance of Theorem 3. In particular, let F be specified by

$$F(k, r) = k + r \quad \text{for } (k, r) \in \mathbb{R}^2_+, \tag{10}$$

and let $(k_0, m_0) = (1, 1)$ be the vector of initial stocks of capital and resource. Even though assumption **A2** and most of assumption **A3** hold, both **A1** (since $F(k, r) = 0$ requires that both inputs are zero) and the last part of **A3** (since $F_{22} = 0$) are violated. Hence, maximin existence is not guaranteed by Theorem 3. And indeed, as demonstrated in the appendix, $C(1, 1) = (0, 2)$, implying that any consumption level below 2 can be sustained indefinitely. However, since the resource stock cannot be instantaneously transformed into capital, it is not feasible to maintain a level of consumption that never falls below 2. Thus, there is no maximin path in this model.

4 Proving an alternative interpretation of Hartwick's rule

We break up the proof of Lemma 1 into several steps.

Step 1: There exists a solution to (5). Since $(c, k) \in D$, there is some $r_0 > 0$ such that $F(k, r_0) > c$. Since $F(k, r)$ is continuous and increasing in r with $F(k, 0) - c < 0$ and $F(k, r_0) - c > 0$, there is a unique $\underline{r} \in (0, r_0)$ such that $F(k, \underline{r}) = c$. Define:

$$R' := \left\{ r \geq \underline{r} : \frac{F(k, r) - c}{r} \geq \frac{F(k, r_0) - c}{r_0} \right\} .$$

Then $r_0 \in R'$, so that R' is non-empty. Since $(F(k, r) - c)/r$ is a continuous function of r on $[\underline{r}, \infty)$, the set R' is closed. It follows from $F(k, 0) = 0$ and the concavity of F that $F(k, r) \leq F(k/r, 1)r$ if $r > 1$. Hence, by $F(0, r) = 0$ and the continuity of F,

$$\lim_{r\to\infty} \frac{F(k, r) - c}{r} \leq \lim_{r\to\infty} \frac{F(k, r)}{r} \leq \lim_{r\to\infty} F\left(\tfrac{k}{r}, 1\right) = 0 ,$$

implying that there exists $\bar{r} > 0$ such that $r \notin R'$ for $r > \bar{r}$. Hence, $R' \subset [\underline{r}, \bar{r}]$, so that R' is bounded. As $(F(k, r) - c)/r$ is a continuous function of r on R', there exists a solution to $\max_{r\in R'}(F(k, r) - c)/r$. By the definition of $\underline{r}$, this is also a solution to (5).

Step 2: There is at most one solution to (HaR). Define:

$$V(r) \equiv F(k, r) - c - F_2(k, r)r \quad \text{for all } r > 0 \tag{11}$$

Note that V is a C^1 function on $\mathbb{R}_{++}$.

Suppose r' and r'' were both solutions to (HaR), with $0 < r' < r''$. Then $V(r') = V(r'') = 0$. We can find $0 < a < r'$ and $b > r''$, and define

$$U(r) \equiv \int_a^r V(s)ds \quad \text{for } r \in [a, b] \,.$$

Then, we have $U'(r) = V(r)$ for all $r \in (a, b)$.[5] By using (11), we can infer that U is a C^2 function on (a, b), and for all $r \in (a, b)$, $U''(r) = V'(r) = -F_{22}(k, r)r > 0$. Thus, U is a strictly convex C^2 function on (a, b), and we get the contradiction:

$$0 = U'(r')(r'' - r') < U(r'') - U(r') < U'(r'')(r'' - r') = 0 \,.$$

Step 3: There is a unique solution to (5), and this uniquely solves (HaR). Since $(F(k, r) - c)/r$ is a continuously differentiable function of r on $\mathbb{R}_{++}$, any solution r to (5) satisfies the first-order condition

$$\frac{F_2(k, r)}{r} - \frac{F(k, r) - c}{r^2} = 0 \,,$$

and therefore is also a solution to (HaR). By Steps 1 and 2, there is a unique solution to (5). We denote the unique solution to (5) by $\mathbf{r}(c, k)$. Then, $\mathbf{r}(c, k)$ is a solution to (HaR), and is the only solution to (HaR) by Step 2.

Step 4: The unique solution to (5), $\mathbf{r}(c, k)$, is continuously differentiable on D. Note that by continuity of F, the set D is open in $\mathbb{R}^2$. Define $Y := D \times \mathbb{R}_{++}$. Clearly, Y is an open set in $\mathbb{R}^3$, and we can define:

$$H(c, k, r) = F(k, r) - c - F_2(k, r)r \quad \text{for } (c, k, r) \in Y \,.$$

Then, H is continuously differentiable on Y, and by Step 3, we have:

$$H(c, k, r) = 0 \quad \text{for } r = \mathbf{r}(c, k) \,.$$

Furthermore, for $r = \mathbf{r}(c, k)$,

$$\frac{\partial H(c, k, r)}{\partial r} = -F_{22}(k, r)r > 0 \,.$$

Thus, by the implicit function theorem,[6] there is an open set $N \subset X := (0, c') \times (k', \infty)$ containing (c, k) and an open set $M \subset Y$, containing $(c, k, \mathbf{r}(c, k))$, and a *unique* function $g : N \to \mathbb{R}_{++}$, such that

[5] See Rudin (1976), Theorem 6.20, p. 133.

[6] See Rudin (1976), Theorem 9.28, p. 224–225.

(i) for all $(\tilde{c}, \tilde{k}) \in N$, we have $H(\tilde{c}, \tilde{k}, g(\tilde{c}, \tilde{k})) = 0$, and

(ii) $g(c, k) = \mathbf{r}(c, k)$.

Furthermore, g is continuously differentiable on N. Since, by Step 3, we certainly have $H(\tilde{c}, \tilde{k}, \mathbf{r}(\tilde{c}, \tilde{k})) = 0$ for all $(\tilde{c}, \tilde{k}) \in N \subset D$, we can infer that $g(\tilde{c}, \tilde{k}) = \mathbf{r}(\tilde{c}, \tilde{k})$ for all $(\tilde{c}, \tilde{k}) \in N$. Thus, $\mathbf{r}$ is continuously differentiable on N. Since $(c, k) \in D$ was arbitrary, $\mathbf{r}$ is continuously differentiable on D.

Step 5: $\mathbf{r}_1(c, k)$ is positive. By definition of $\mathbf{r}$ on D, we have:

$$F(k, \mathbf{r}(c, k)) - c - F_2(k, \mathbf{r}(c, k))\mathbf{r}(c, k) = 0 \quad \text{for all } (c, k) \in D\,.$$

Thus, differentiating this equation w.r.t. c yields

$$F_2(k, \mathbf{r}(c, k))\mathbf{r}_1(c, k) - 1 - F_{22}(k, \mathbf{r}(c, k))\mathbf{r}_1(c, k)\mathbf{r}(c, k) - F_2(k, \mathbf{r}(c, k))\mathbf{r}_1(c, k) = 0\,,$$

from which (6) can be inferred.

5 Proving the sufficiency theorem

We first provide some preliminary results in the form of two lemmas, before providing the proof of Theorem 1.

Lemma 2 *Let $(c, k_0) \in D$. Consider the differential equation:*

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c\,; \quad x(0) = k_0\,, \tag{12}$$

where $\mathbf{r} : D \to \mathbb{R}_{++}$ is the function obtained in Lemma 1. If $k^c(t)$ is a solution to the differential equation for $t \in [0, T)$, for some $T > 0$, then

(i) $k^c(t) \geq k_0$ for all $t \in [0, T)$,

(ii) $\dot{k}^c(t) > 0$ for all $t \in [0, T)$ and $k^c(t)$ is monotonically increasing on $[0, T)$.

Proof. Since $(c, k_0) \in D$ implies that $F(k_0, \mathbf{r}(c, k_0)) - c > 0$, we have $\dot{k}^c(0) > 0$. Since $\dot{k}^c$ is continuous, there is $\varepsilon \in (0, T)$ such that $\dot{k}^c(t) > 0$ for all $t \in [0, \varepsilon]$, and so by the Mean Value theorem, $k^c(t) > k_0$ for all $t \in (0, \varepsilon]$. We claim that $k^c(t) \geq k_0$ for all $t \in [0, T)$. If not, there is $\tau' \in (\varepsilon, T)$ such that $k^c(\tau') < k_0$. Let $\tau \equiv \inf\{t \in (\varepsilon, T) : k^c(t) < k_0\}$. Then, $\varepsilon \leq \tau \leq \tau' < T$, and $k^c(\tau) \leq k_0$ by continuity of k^c. Also, by definition of τ, $k^c(t) \geq k_0$ for all $t \in [0, \tau)$. Thus,

$k^c(\tau) \geq k_0$ by continuity of k^c, and consequently $k^c(\tau) = k_0$. Then, using the fact that $k^c(t) \geq k^c(\tau)$ for all $t \in [0, \tau)$, we must have $\dot{k}^c(\tau) \leq 0$. On the other hand, since $k^c(\tau) = k_0$, we must have $\dot{k}^c(\tau) > 0$ by using (12). This contradiction establishes our claim and hence (i).

Since $(c, k_0) \in D$ and $k^c(t) \geq k_0$ for all $t \in [0, t)$, it follows that $(c, k^c(t)) \in D$ for all $t \in [0, T)$. Thus, by definition of $\mathbf{r}$, we must have $\dot{k}^c(t) > 0$ for all $t \in [0, T)$, and so $k^c(t)$ is monotonically increasing on $[0, T)$ by the Mean Value Theorem. This establishes (ii). ∎

Remark 1 Under the hypothesis of Lemma 2, we have $k^c(t)$ monotonically increasing on $[0, T)$. Then, either $k_c(t)$ is bounded above on $[0, T)$, in which case a finite limit, $\lim_{t \to T} k^c(t)$, exists; or, $k_c(t)$ is not bounded above on $[0, T)$, in which case $k^c(t) \to \infty$ as $t \to T$. In either case, we define

$$k^c(T) = \lim_{t \to T} k^c(t)\,,$$

it being understood that the limit above belongs to $(k_0, \infty]$.

Lemma 3 *Let $(c, k_0) \in D$. Suppose that there is a solution $k^c(t)$ to the differential equation*

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c\,; \quad x(0) = k_0 \tag{12}$$

for $t \in [0, T)$ for some $T > 0$, where $\mathbf{r} : D \to \mathbb{R}_{++}$ is the function obtained in Lemma 1. Then, for every $T' \in (0, T)$,

$$\int_0^{T'} \mathbf{r}(c, k^c(t))dt = \int_{k_0}^{k^c(T')} \frac{1}{F_2(x, \mathbf{r}(c, x))}dx \tag{13}$$

and

$$\lim_{T' \to T} \int_0^{T'} \mathbf{r}(c, k^c(t))dt = \int_{k_0}^{k^c(T)} \frac{1}{F_2(x, \mathbf{r}(c, x))}dx\,. \tag{14}$$

Furthermore, if:

$$\infty > S := \lim_{T' \to T} \int_0^{T'} \mathbf{r}(c, k^c(t))dt \tag{15}$$

then $k^c(T) < \infty$.

Proof. For every $T' \in (0, T)$, $\mathbf{r}(c, k^c(t))$ is continuous on $[0, T']$ and the Riemann integral $\int_0^{T'} \mathbf{r}(c, k^c(t))dt$ is well-defined. Define:

$$f(x) = \frac{1}{F_2(x, \mathbf{r}(c, x))} \text{ for } x \in [k_0, k^c(T')] \quad \text{and} \quad g(t) = k^c(t) \text{ for } t \in [0, T']\,.$$

Then, by the change of variable formula,[7]

$$\int_{k_0}^{k^c(T')} f(x)dx = \int_0^{T'} f(g(t))g'(t)dt$$

so that

$$\int_{k_0}^{k^c(T')} \frac{1}{F_2(x, \mathbf{r}(c,x))} dx = \int_0^{T'} \left[\frac{1}{F_2(k^c(t), \mathbf{r}(c, k^c(t)))}\right] \dot{k}^c(t)dt\,.$$

Thus, (13) follows from (HaR), which is satisfied by definition of the function $\mathbf{r}$ and Lemma 1. Furthermore, (14) follows by letting $T' \to T$ in (13).

Assume now that (15) holds. Write $r^c(t) = \mathbf{r}(c, k^c(t))$ for $t \in [0,T)$ and

$$\lambda = \max\{F(1, 1/k_0)/k_0, F(k_0, 1), F(k_0, 1)/k_0\}\,.$$

For $t \in [0,T)$, either (i) $r^c(t) \le k^c(t)/k_0$, or (ii) $r^c(t) > k^c(t)/k_0$. Case (i) can be divided into two subcases: (i)(a) $k^c(t) \le 1$ and (i)(b) $k^c(t) > 1$. In case (i)(a),

$$\dot{k}^c(t)/k^c(t) \le F(k^c(t), r^c(t))/k^c(t) \le F(1, 1/k_0)/k_0 \le \lambda\,. \tag{16}$$

In case (i)(b),

$$\dot{k}^c(t)/k^c(t) \le F(k^c(t), r^c(t))/k^c(t) \le F(1, r^c(t)/k^c(t)) \le F(1, 1/k_0) \le \lambda\,. \tag{17}$$

In case (ii), we have that $r^c(t) > k^c(t)/k_0 \ge 1$, so

$$\dot{k}^c(t) \le F(k^c(t), r^c(t)) \le r^c(t)F(k^c(t)/r^c(t), 1) \le r^c(t)F(k_0, 1)$$

and

$$\dot{k}^c(t)/k^c(t) \le r^c(t)F(k_0, 1)/k_0 \le \lambda r^c(t)\,. \tag{18}$$

Let $\Lambda = \{t \in [0,T) : r^c(t) > k^c(t)/k_0\}$. Then, by (16)–(18), we have:

$$\int_{[0,T)} \left(\dot{k}^c(t)/k^c(t)\right) dt \le \int_\Lambda \lambda r^c(t)dt + \int_{[0,T)\setminus\Lambda} \lambda dt \le \lambda(S+T)\,. \tag{19}$$

Thus, for every $t \in (0,T)$, $\ln k^c(t) \le \ln k_0 + \lambda(S+T)$, showing that $k^c(T) < \infty$. ■

We now present the proof of Theorem 1.

Proof of Theorem 1. Let $(k_0, m_0) >> 0$ be given. Then, since (7) is satisfied, we can find $c \in D(k_0)$ such that

$$\int_{k_0}^{\infty} \frac{1}{F_2(x, \mathbf{r}(c,x))} dx \le m_0\,. \tag{20}$$

[7] See Apostol (1974) Theorem 7.36, p. 164.

Note that since $c \in D(k_0)$, we can find $k_0 > \varepsilon > 0$ such that $c \in D(k)$ for all $k \in (k_0 - \varepsilon, \infty)$. Then $\mathbf{r}(c, k)$ is a C^1 function of k from the open set $(k_0 - \varepsilon, \infty)$ to $\mathbb{R}_{++}$. Thus, by defining

$$f(k) = F(k, \mathbf{r}(c, k)) - c \quad \text{for all } k \in (k_0 - \varepsilon, \infty),$$

we see that f is a C^1 function from the open set $(k_0 - \varepsilon, \infty)$ to $\mathbb{R}$. Using Theorem 1 of Hirsch and Smale (1974, pp. 162–163), there is $a > 0$ and a unique solution $x^c : [0, a) \to (k_0 - \varepsilon, \infty)$ to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, a)$$

satisfying $x^c(0) = k_0$. Define the set B of $b \in [a, \infty]$ such that there is a solution $\phi^c : [0, b) \to (k_0 - \varepsilon, \infty)$ to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, b)$$

satisfying $\phi^c(0) = k_0$. Since $a \in B$, the set B is non-empty, and we define $\beta = \sup B$.

Define $a' = a/2$. For each $b \in [0, \beta)$, we can find $b' \in (b, \beta)$ and some solution $\psi : [0, b') \to (k_0 - \varepsilon, \infty)$ to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, b')$$

satisfying $\psi(0) = k_0$. Since $x^c(t)$ is the *unique* solution to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, a)$$

we also have $\psi(a') = x^c(a')$. Furthermore, if there is $b'' \in (b, \beta)$ and some solution $\xi : [0, b'') \to (k_0 - \varepsilon, \infty)$ to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, b'')$$

satisfying $\xi(0) = k_0$, then $\xi(a') = x^c(a') = \psi(a')$. By letting $\bar{b} := \min\{b', b''\}$ and using the Lemma in Hirsch and Smale (1974, p. 171), it follows that $\xi(t) = \psi(t)$ for all $t \in (0, \bar{b})$ and therefore for all $t \in [0, \bar{b})$. Then, we can define

$$k^c(b) = \psi(b).$$

This well-defines the function k^c from $[0, \beta)$ to $(k_0 - \varepsilon, \infty)$. In particular, this definition implies that $k^c(t) = \psi(t)$ for all $t \in [0, b]$. Then $k^c : [0, \beta) \to (k_0 - \varepsilon, \infty)$ is a solution to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, \beta)$$

since for every $t \in [0, \beta)$, it agrees with a solution; furthermore, $k^c(0) = k_0$. It follows that if $\kappa^c : [0, \alpha) \to (k_0 - \varepsilon, \infty)$ is any solution to

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, \alpha)$$

satisfying $\kappa(0) = k_0$, then $\alpha \leq \beta$. The interval $[0, \beta)$ is called the *maximal right interval* of existence, given the initial condition. By Lemma 2, $\dot{k}^c(t) > 0$ and $k^c(t) \in [k_0, \infty)$ for all $t \in [0, \beta)$.

We now claim that

$$\beta = \infty \,. \tag{21}$$

For all $T' \in (0, \beta)$, using (13) in Lemma 3 and (20), we have:

$$\int_0^{T'} \mathbf{r}(c, k^c(t))dt = \int_{k_0}^{k^c(T')} \frac{1}{F_2(x, \mathbf{r}(c, x))} dx \leq \int_{k_0}^{\infty} \frac{1}{F_2(x, \mathbf{r}(c, x))} dx \leq m_0 \,.$$

Thus,

$$\infty > m_0 \geq S \equiv \lim_{T' \to \beta,} \int_0^{T'} \mathbf{r}(c, k^c(t))dt \tag{22}$$

and by Lemma 3, we have $k^c(\beta) < \infty$, where $k^c(\beta) \equiv \lim_{t \to \beta} k^c(t)$. Then by using the Theorem in Hirsch and Smale (1974, p. 171), claim (21) is established.

Given (21), we know that k^c from $[0, \infty)$ to $(k_0 - \varepsilon, \infty)$ is a solution to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, \infty)$$

satisfying $k^c(0) = k_0$. Define $c^c(t) = c$ and $r^c(t) = \mathbf{r}(c, k^c(t))$ for $t \in [0, \infty)$. It follows from (22) that $(c^c(t), k^c(t), r^c(t))$ is a path from (k_0, m_0). Furthermore, for all $t \geq 0$, $c^c(t) = c$ and so $C(k_0, m_0)$ is non-empty. ■

6 Proving the necessity theorem

We first provide some preliminary results in the form of two lemmas, before providing the proof of Theorem 2.

The phase diagram argument illustrated in Figure 1 is based on paths in (k, m) space where the stock of augmentable capital k is a function of the remaining resource stock m. Interior paths have this property, while non-interior paths where resource extraction is zero at some time—so that consumption comes from divestment of augmentable capital—do not. This motivates the following lemma.

Lemma 4 *Assume that* $C(k_0, m_0)$ *is non-empty, and let* $c \in \text{int}(C(k_0, m_0))$*. Then there is an interior path* $(c(t), k(t), r(t))$ *from* (k_0, m_0) *with* $c(t) > c$ *for all* $t \geq 0$.

Proof. Since $c \in \text{int}(C(k_0, m_0))$, there is $\lambda \in (0, 1)$ such that $c = \lambda c'$ and $c' \in C(k_0, m_0)$. By the definition of $C(k_0, m_0)$, there is $(c'(t), k'(t), r'(t))$ from (k_0, m_0) with $c'(t) \geq c'$ for $t \geq 0$. Construct $(c(t), k(t), r(t))$ as follows:

$$\begin{aligned} c(t) &= F(k(t), r(t)) - \dot{k}(t) && \text{for all } t \geq 0 \\ k(t) &= (1-\lambda)k_0 + \lambda k'(t) && \text{for all } t \geq 0 \\ r(t) &= \lambda[r'(t) + (\epsilon/e^t)] && \text{for all } t \geq 0\,, \end{aligned} \tag{23}$$

where $\epsilon = (1-\lambda)m_0/\lambda > 0$. We must show that $(c(t), k(t), r(t))$ is an interior path from (k_0, m_0) with $c(t) > c$ for $t \geq 0$.

Clearly, $k(t)$ is a differentiable function of t, with $\dot{k}(t) = \lambda \dot{k}'(t)$ for $t \geq 0$, and $(c(t), r(t))$ are continuous functions of t. Using (23), for $t \geq 0$,

$$\begin{aligned} c(t) = F(k(t), r(t)) - \dot{k}(t) &> F(\lambda k'(t), r'(t)) - \lambda \dot{k}'(t) \\ &\geq \lambda[F(k(t), r(t)) - \dot{k}(t)] = \lambda c'(t) \geq \lambda c' = c\,. \end{aligned}$$

Again using (23), $r(t) > 0$ for $t \geq 0$ and

$$\int_0^t r(\tau)d\tau \leq \lambda \int_0^t r'(\tau)d\tau + \lambda\epsilon$$

so that:

$$\int_0^t r(\tau)d\tau \leq \lambda \int_0^t r'(\tau)d\tau + \lambda\epsilon < \lambda m_0 + (1-\lambda)m_0 = m_0\,.$$

Thus, (3) is satisfied. Also $k(t) \geq (1-\lambda)k_0 > 0$ for $t \geq 0$, and $k(0) = (1-\lambda)k_0 + \lambda k_0 = k_0$ so (2) is satisfied. ■

We also need to establish that positive consumption cannot be sustained if the stock of augmentable capital is bounded above.

Lemma 5 *If a path* $(c(t), k(t), r(t))$ *from* (k_0, m_0) *has the property that* $c(t) \geq c > 0$ *for* $t \geq 0$*, then* $\limsup_{t\to\infty} k(t) = \infty$.

Proof. Suppose on the contrary that there is $\bar{k} \in (0, \infty)$ such that $k(t) \leq \bar{k}$ for $t \geq 0$. Then $F(\bar{k}, 0) = 0$ and $F(\bar{k}, \cdot)$ is continuous, concave and increasing on $\mathbb{R}_+$. Using Jensen's inequality, we have for all $T > 0$:

$$\frac{1}{T}\int_0^T F\left(\bar{k}, r(t)\right) dt \leq F\left(\bar{k}, \frac{1}{T}\int_0^T r(t)dt\right) \leq F\left(\bar{k}, \frac{m_0}{T}\right)\,.$$

Then we get:

$$k(T) - k_0 = \int_0^T \dot{k}(t)dt \leq \int_0^T F(\bar{k}, r(t))dt - Tc \leq T\left[F\left(\bar{k}, \frac{m_0}{T}\right) - c\right].$$

Since $\lim_{T\to\infty} F\left(\bar{k}, m_0/T\right) = 0$, this implies $k(T) < 0$ for large T, contradicting that $k(t) \geq 0$ for $t \geq 0$. ■

We now provide the proof of Theorem 2.

Proof of Theorem 2. Fix $(c(t), k(t), r(t))$ from (k_0, m_0) with $c(t) > c > 0$ for $t \geq 0$ established in Lemma 4. By Lemma 5, $\limsup_{t\to\infty} k(t) = \infty$. Let $T_0 = \inf\{t \geq 0 : k(t) > k_0\}$. Then $k(T_0) = k_0$ and $\dot{k}(T_0) \geq 0$ so that

$$F(k_0, r(T_0)) = \dot{k}(T_0) + c(T_0) \geq c(T_0) > c\,,$$

establishing that $(k_0, c) \in D$ and $c \in D(k_0)$. So, we can find $k_0 > \varepsilon > 0$ such that $c \in D(k)$ for all $k \in (k_0 - \varepsilon, \infty)$. Then $\mathbf{r}(c, k)$ is a C^1 function of k from the open set $(k_0 - \varepsilon, \infty)$ to $\mathbb{R}_{++}$. Thus, defining

$$f(k) = F(k, \mathbf{r}(c, k)) - c \quad \text{for all } k \in (k_0 - \varepsilon, \infty)\,,$$

we see that f is a C^1 function from the open set $(k_0 - \varepsilon, \infty)$ to $\mathbb{R}$. Using Theorem 1 of Hirsch and Smale (1974, pp. 162–163), there is $a > 0$ and a unique solution $x^c : [0, a) \to (k_0 - \varepsilon, \infty)$ to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, a)\,.$$

satisfying $x^c(0) = k_0$.

Using exactly the method used in the proof of Theorem 1, one can show that there is an interval $[0, \beta)$ and a solution $k^c : [0, \beta) \to (k_0 - \varepsilon, \infty)$, to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, \beta)$$

satisfying $k^c(0) = k_0$, such that if $\phi^c : [0, \alpha) \to (k_0 - \varepsilon, \infty)$ is a solution to

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c \quad \text{for } t \in [0, \alpha)$$

satisfying $\phi^c(0) = k_0$, then $\alpha \leq \beta$. By Lemma 2, $\dot{k}^c(t) > 0$ and $k^c(t) \in [k_0, \infty)$ for all $t \in [0, \beta)$.

We now proceed to verify that (20) holds. It is sufficient to establish that $\int_{k_0}^{k_1}(1/F_2(x, \mathbf{r}(c, x)))dx < m_0$ for all $k_1 > k_0$. Suppose on the contrary that there is $k_1' \in (k_0, \infty)$ such that $\int_{k_0}^{k_1'} (1/F_2(x, \mathbf{r}(c,x)))dx \geq m_0$. Then, there is $k_1 \in (k_0, k_1']$ such that

$$\int_{k_0}^{k_1} \frac{1}{F_2(x, \mathbf{r}(c,x))} dx = m_0 \, .$$

We claim now that

$$k^c(t) > k_1 \text{ for some } t \in [0, \beta) \, . \tag{24}$$

If $\beta < \infty$, then this follows directly from the Theorem of Hirsch and Smale (1974, p. 171). If $\beta = \infty$, and claim (24) does not hold, then $k^c(t) \leq k_1$ for all $t \geq 0$. By Lemma 3, we have

$$\int_0^\infty \mathbf{r}(c, k^c(t))dt = \int_{k_0}^{k_\infty^c} \frac{1}{F_2(x, \mathbf{r}(c,x))} dx \leq \int_{k_0}^{k_1} \frac{1}{F_2(x, \mathbf{r}(c,x))} dx = m_0 \, ,$$

where $k_\infty^c := \lim_{t\to\infty} k^c(t)$. However, this means that $(c, k^c(t), \mathbf{r}(c, k^c(t)))$ is a path from (k_0, m_0), and since $c > 0$, it follows from Lemma 5 that $\limsup_{t\to\infty} k^c(t) = \infty$. This clearly contradicts the hypothesis that claim (24) does not hold. Thus, in either case, claim (24) is valid.

Using (24), we infer that there is $T^* \in (0, \beta)$ such that $k^c(T^*) = k_1$ and so by Lemma 3,

$$\int_0^{T^*} \mathbf{r}(c, k^c(t))dt = \int_{k_0}^{k_1} \frac{1}{F_2(x, \mathbf{r}(c,x))} dx = m_0 \, .$$

Write $r^c(t) = \mathbf{r}(c, k^c(t))$ and $m^c(t) = m_0 - \int_0^t r^c(\tau)d\tau$ for $t \in [0, T^*]$. Since $m^c : [0, T^*] \to [0, m_0]$ is continuously differentiable and decreasing on $[0, T^*]$, it has an inverse function $i^c : [0, m_0] \to [0, T^*]$ which is continuously differentiable and decreasing on $[0, m_0]$. We define $h^c : [0, m_0] \to [k_0, k_1]$ by $h^c(m) = k^c(i^c(m))$. Then h^c is a continuously differentiable and decreasing function on $[0, m_0]$ which determines the stock of augmentable capital as a function of the remaining resource stock when the differential equation (12) is satisfied.

As $(c(t), k(t), r(t))$ from (k_0, m_0) is interior, $m : [0, \infty) \to (0, m_0]$ is continuously differentiable and decreasing on $[0, \infty)$ and has an inverse function $i : (0, m_0] \to [0, \infty)$ which is continuously differentiable and decreasing on $(0, m_0]$. We define $h : (0, m_0] \to [k_0, \infty)$ by $h(m) = k(i(m))$. Then h is a continuously differentiable function on $(0, m_0]$ (but not necessarily a decreasing function as $\dot{k}(t)$ is not necessarily positive) which determines the stock of augmentable capital as a function of the remaining resource stock when $(c(t), k(t), r(t))$ is followed.

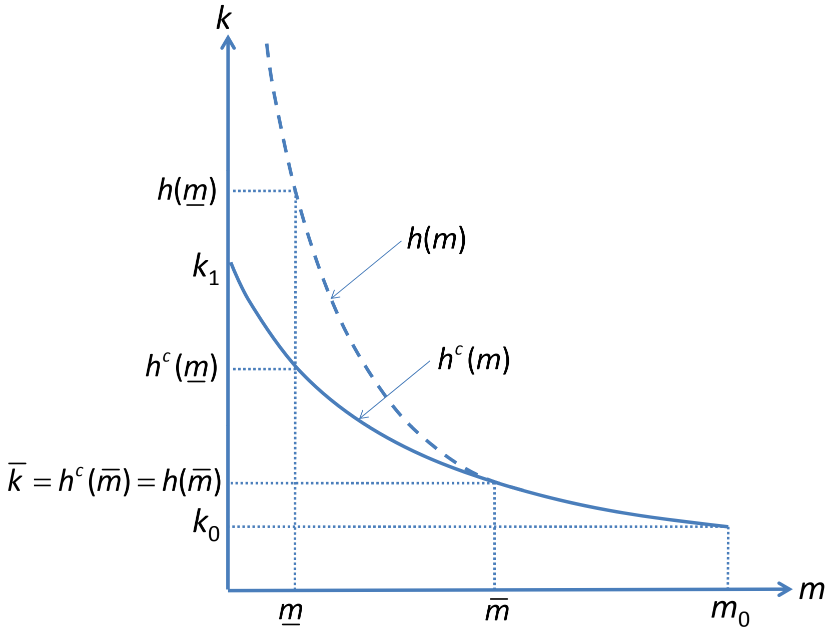

Figure 2: Illustration of the proof of Theorem 2

By Lemma 5 there is $T' \in (0,\infty)$ such that $k(T') > k_1$. Hence,

$$h(\underline{m}) = k(T') > k_1 = k^c(T^*) = h^c(0) > h^c(\underline{m}) \quad \text{where } \underline{m} = m(T') \in (0, m_0)\,,$$

while $h(m_0) = k_0 = h^c(m_0)$ so that $\{m \in [\underline{m}, m_0] : h(m) \le h^c(m)\}$ is non-empty. By continuity of h and h^c on $(0, m_0]$, $\bar{m} = \inf\{m \in [\underline{m}, m_0] : h(m) \le h^c(m)\} > \underline{m}$ and $h(\bar{m}) = h^c(\bar{m})$; let $\bar{k}$ denote this common value. This is illustrated in Figure 2.

Since $h(m) > h^c(m)$ for $m \in (\underline{m}, \bar{m})$ and $h(\bar{m}) = h^c(\bar{m})$, we have

$$\frac{h(m) - h(\bar{m})}{m - \bar{m}} < \frac{h^c(m) - h^c(\bar{m})}{m - \bar{m}}$$

for all $m \in (\underline{m}, \bar{m})$. Letting $m \to \bar{m}$ and noting that h and h^c are continuously differentiable on $(0, m_0]$, we obtain $h'(\bar{m}) \le h^{c\prime}(\bar{m})$, and equivalently:

$$(-h'(\bar{m})) \ge -(h^{c\prime}(\bar{m}))\,. \tag{25}$$

Since $\bar{m} \in (0, m_0)$, there is a unique T such that $m(T) = \bar{m}$ and

$$h'(\bar{m})\dot{m}(T) = h'(m(T))\dot{m}(T) = \dot{k}(T)\,.$$

As $\dot{m}(T) = -r(T) < 0$ and $\dot{k}(T) = F(\bar{k}, r(T)) - c(T)$, we have that

$$-h'(\bar{m}) = \frac{\dot{k}(T)}{-\dot{m}(T)} = \frac{F(\bar{k}, r(T)) - c(T)}{r(T)}\,. \tag{26}$$

Since $\bar{m} \in (0, m_0)$, there is a unique T^c such that $m^c(T^c) = \bar{m}$ and

$$h^{c\prime}(\bar{m})\dot{m}^c(T^c) = h^{c\prime}(m^c(T^c))\dot{m}^c(T^c) = \dot{k}^c(T^c)\,.$$

As $\dot{m}^c(T^c) = -r^c(T^c) < 0$ and $\dot{k}^c(T) = F(\bar{k}, r^c(T)) - c$, we have that

$$-h^{c\prime}(\bar{m}) = \frac{\dot{k}^c(T^c)}{-\dot{m}^c(T^c)} = \frac{F(\bar{k}, r^c(T^c)) - c}{r^c(T^c)}\,. \tag{27}$$

Recall that $c(T) > c$. Hence, it follows from Lemma 1 that

$$\frac{F(\bar{k}, r(T)) - c(T)}{r(T)} < \frac{F(\bar{k}, r(T)) - c}{r(T)} \leq \frac{F(\bar{k}, \mathbf{r}(c, \bar{k})) - c}{\mathbf{r}(c, \bar{k})} = \frac{F(\bar{k}, r^c(T^c)) - c}{r^c(T^c)}\,.$$

Combined with (26) and (27) this contradicts (25). Thus, (20) must hold.

Now, pick any $c' \in (0, c)$. Note that since $c \in D(k_0)$, we have $c' \in D(k_0)$. By Lemma 1, $\mathbf{r}_1 > 0$ on D, and we have $\mathbf{r}(c', x,) < \mathbf{r}(c, x)$ for all $x \in [k_0, \infty)$. Therefore, since $F_{22} < 0$ on $\mathbb{R}^2_{++}$, we have:

$$F_2(x, \mathbf{r}(c', x)) > F_2(x, \mathbf{r}(c, x)) \quad \text{for all } x \in [k_0, \infty)\,. \tag{28}$$

Using (28) in (20), we get:

$$\int_{k_0}^{\infty} \frac{1}{F_2(x, \mathbf{r}(c', x))} dx < m_0\,. \tag{29}$$

Using (29) and $c' \in D(k_0)$, we have that (7) holds. This establishes the theorem. ■

7 Proving the maximin existence theorem

By the remark prior to Theorem 3, it is sufficient to show the following proposition:

Proposition 1 *If $C(k_0, m_0)$ is non-empty, then there is an egalitarian maximin path $(c^*(t), k^*(t), r^*(t))$ from (k_0, m_0) satisfying $c^*(t) = \sup C(k_0, m_0)$ and (HaR) for all $t \geq 0$.*

We first note the existence of a Hartwick path from (k_0, m_0) given c if there is a path from (k_0, m_0) that sustains consumption above c.

Proposition 2 *If $C(k_0, m_0)$ is non-empty and $c \in \text{int}(C(k_0, m_0))$, then (20) holds and there is a path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) satisfying $c^c(t) = c$ and (HaR) for all $t \geq 0$.*

Proof. The proof of Theorem 2 shows that (20) holds and establishes existence of path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) satisfying $c^c(t) = c$ and (HaR) for all $t \geq 0$, provided that $\beta = \infty$. We have that $\beta = \infty$ by the argument in the penultimate paragraph of the proof of Theorem 1. ■

We then provide some preliminary results in the form of three lemmas, before providing the proof of Proposition 1. The two first lemmas show that (i) the set of consumptions levels c that can be sustained from (k_0, m_0) is bounded above and that (ii) this set's least upper bound allows for positive capital accumulation from the initial stock k_0.

Lemma 6 *If $C(k_0, m_0)$ is non-empty, then $C(k_0, m_0)$ is bounded.*

Proof. By Proposition 2, if $c \in \text{int}(C(k_0, m_0))$, then there is a path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) satisfying $c^c(t) = c$ and (HaR) for all $t \geq 0$. It suffices to show that, for any $c \in \text{int}(C(k_0, m_0))$, $c < F(\bar{k}, m_0)$, where $\ln \bar{k} = \ln k_0 + \lambda(m_0 + 1)$ and

$$\lambda = \max\{F(1, 1/k_0)/k_0, F(k_0, 1), F(k_0, 1)/k_0\} \, .$$

By using (14) in Lemma 3 and (20) and repeating the last part of the proof of Lemma 3, $\ln k^c(t) \leq \ln k_0 + \lambda(m_0 + t)$ for all $t \geq 0$. Hence, $k^c(t) \leq \bar{k}$ for $t \in [0, 1]$. Then $F(\bar{k}, 0) = 0$ and $F(\bar{k}, \cdot)$ is continuous, concave and increasing on $\mathbb{R}_+$. Using Jensen's inequality, we have:

$$\int_0^1 F\left(\bar{k}, r^c(t)\right) dt \leq F\left(\bar{k}, \int_0^1 r^c(t) dt\right) \leq F\left(\bar{k}, m_0\right) \, .$$

Then we get:

$$k^c(1) - k_0 = \int_0^1 \dot{k}^c(t) dt \leq \int_0^1 F(\bar{k}, r^c(t)) dt - c \leq F\left(\bar{k}, m_0\right) - c \, .$$

Since $k^c(1) > k_0$, it follows that $c < F(\bar{k}, m_0)$. ■

Lemma 7 *If $C(k_0, m_0)$ is non-empty, then $\sup C(k_0, m_0) \in D(k_0)$.*

Proof. Let $\bar{c} := \sup C(k_0, m_0)$. Suppose on the contrary that $\lim_{r \to \infty} F(k_0, r) \leq \bar{c}$. We consider two cases separately: (i) $\lim_{r \to \infty} F(k_0, r) < \bar{c}$; $\lim_{r \to \infty} F(k_0, r) = \bar{c}$.

In case (i), it follows from Proposition 2 that there is a path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) satisfying $c^c(t) = c$ and (HaR) for all $t \geq 0$, for which $\lim_{r \to \infty} F(k_0, r) < c < \bar{c}$. Since $\dot{k}^c(0) = F(k_0, r^c(0)) - c > 0$, we obtain the following contradiction:

$$c < F(k_0, r^c(0)) \leq \lim_{r \to \infty} F(k_0, r) < c \, .$$

In case (ii), we have $F(k_0, r) < \bar{c}$ for all $r \geq 0$, which implies:

$$F\big(k_0, \tfrac{\bar{c}+k_0}{k_0} m_0\big) < \bar{c}\,.$$

Consequently, there is $\theta > 1$ such that

$$\underline{c} := F\big(\theta k_0, \tfrac{\bar{c}+k_0}{k_0} m_0\big) < \bar{c}\,. \tag{30}$$

It follows from Proposition 2 that there is a path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) satisfying $c^c(t) = c$ and (HaR) for all $t \geq 0$, with $\underline{c} < c < \bar{c}$ and $\bar{c} - c < (\theta - 1)k_0$.

Since $k^c(0) = k_0 < \theta k_0$ and $\dot{k}^c(t) > 0$ for all $t \in [0, \infty)$, by Lemma 5 there is a unique $T > 0$ such that $k^c(T) = \theta k_0$. For every $t \in [0, T]$, $k^c(t) \leq \theta k_0$, so that:

$$0 < \dot{k}^c(t) = F(k^c(t), r^c(t)) - c \leq F(\theta k_0, r^c(t)) - c\,. \tag{31}$$

This implies that $F(\theta k_0, r^c(t)) > c > \underline{c}$, and using (30) we obtain:

$$r^c(t) > \tfrac{\bar{c}+k_0}{k_0} m_0 \quad \text{for all } t \in [0, T]\,. \tag{32}$$

Also, recalling $\bar{c} - c < (\theta - 1)k_0$, we obtain that, for every $t \in [0, T]$,

$$\begin{aligned} F(\theta k_0, r^c(t)) - c &\leq \theta F(k_0, r^c(t)) - c \leq \theta \bar{c} - c \\ &= (\theta - 1)\bar{c} + (\bar{c} - c) < (\theta - 1)\bar{c} + (\theta - 1)k_0 = (\theta - 1)(\bar{c} + k_0)\,. \end{aligned}$$

By combining this inequality with (31) we obtain

$$(\theta - 1)k_0 = k^c(T) - k^c(0) = \int_0^T \dot{k}^c(t)dt < (\theta - 1)(\bar{c} + k_0)T\,,$$

which yields $T > k_0/(\bar{c} + k_0)$ and, by (32),

$$\int_0^T r^c(t)dt > T \tfrac{\bar{c}+k_0}{k_0} m_0 > \tfrac{k_0}{\bar{c}+k_0} \tfrac{\bar{c}+k_0}{k_0} m_0 = m_0\,.$$

This contradicts that $(c^c(t), k^c(t), r^c(t))$ is a path from (k_0, m_0). ■

The following lemma implies that the least upper bound $\bar{c}$ for consumption levels that can be sustained from (k_0, m_0) can be implemented by a Hartwick path from (k_0, m_0) given $\bar{c}$. This completes the proof of Proposition 1.

Lemma 8 *Assume that $C(k_0, m_0)$ is non-empty, and let $\bar{c} := \sup C(k_0, m_0)$. Then:*

$$\int_{k_0}^{\infty} \frac{1}{F_2(x, \mathbf{r}(\bar{c}, x))} dx \leq m_0\,. \tag{33}$$

Proof. By Lemma 7, $(k_0, \bar{c}) \in D$. By Lemma 1, there are $k' < k_0$ and $c' > \bar{c}$ such that $(0, c') \times (k', \infty) \in D$, and $\mathbf{r}(c, k)$ and $g(c, k) \equiv 1/F_2(k, \mathbf{r}(c, k))$ are continuously differentiable functions on $(0, c') \times (k', \infty)$. We claim that, for every $k_1 > k_0$, $\int_{k_0}^{k_1} g(\bar{c}, x)dx \leq m_0$. Note that since $g(\bar{c}, k)$ is continuous on (k', ∞), the Riemann integral on the left-hand side is well-defined for every $k_1 > k_0$.

Suppose, contrary to our claim, that there is $k_1 > k_0$ such that $\int_{k_0}^{k_1} g(\bar{c}, x)dx > m_0$. Since g is continuous on $(0, c') \times (k', \infty)$, $J(c) \equiv \int_{k_0}^{k_1} g(c, x)dx$ is continuous on the interval $[\bar{c}/2, \bar{c}]$ (see Apostol, 1974, p. 166). In particular, there is $c \in \text{int}(C(k_0, m_0))$ such that $J(c) > m_0$. However, by Proposition 2, if $c \in \text{int}(C(k_0, m_0))$, then (20) holds, leading to a contradiction. ■

Proof of Proposition 1. It follows from Lemma 8 and the proof of Theorem 1 that there is a path $(c^*(t), k^*(t), r^*(t))$ from (k_0, m_0), where k^* from $[0, \infty)$ to $(k_0 - \varepsilon, \infty)$ is a solution to the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(\bar{c}, x(t))) - \bar{c} \quad \text{for } t \in [0, \infty)$$

satisfying $k^*(0) = k_0$, and where $c^*(t) = \bar{c}$ and $r^*(t) = \mathbf{r}(\bar{c}, k^*(t))$ for $t \in [0, \infty)$. The path is egalitarian by construction, maximin by definition of $\bar{c}$, and satisfies (HaR) for all $t \geq 0$ by Lemma 1. ■

8 Proving the maximin efficiency theorem

To discuss the issue of efficiency we first introduce some additional definitions. A path $(c(t), k(t), r(t))$ from (k_0, m_0) is *efficient* if there is no path $(c'(t), k'(t), r'(t))$ from (k_0, m_0) with $c'(t) \geq c(t)$ for all $t \geq 0$ and $c'(\tau) > c(\tau)$ for some $\tau \geq 0$.[8] A path $(c(t), k(t), r(t))$ from (k_0, m_0) is *resource exhausting* if (3) is binding. An interior path $(c(t), k(t), r(t))$ from (k_0, m_0) satisfies *Hotelling's* no-arbitrage *rule* if $r(t)$ is not only continuous but also differentiable and

$$\frac{\dot{F}_2(k(t), r(t))}{F_2(k(t), r(t))} = F_1(k(t), r(t)) \quad \text{for all } t\,. \tag{HoR}$$

An interior path $(c(t), k(t), r(t))$ from (k_0, m_0) satisfying (HoR) satisfies in addition the *capital value transversality condition* if

$$\lim_{t \to \infty} \frac{1}{F_2(k(t), r(t))} k(t) = 0\,. \tag{CVT}$$

[8]Usually we employ the strict inequality for an interval of time, because in continuous time spikes do not matter. Here, however, $c(t)$ is continuous, so that we can use this more stringent definition.

Hotelling's rule is a condition for short-run efficiency, ensuring that for any $T > 0$ it is not feasible to increase consumption on some part of $[0, T]$ without decreasing consumption on some other part of $[0, T]$, for fixed stocks $(k(T), m(T))$ at time T. The capital value transversality condition and resource exhaustion ensure that stocks are not over-accumulated as time goes infinity.

If F satisfies **A1**–**A3** and $C(k_0, m_0)$ is non-empty, then by Propositions 1 and 2, for any $c \in C(k_0, m_0)$, there is a Hartwick path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) satisfying $c^c(t) = c$ for all $t \geq 0$. In particular, there is a maximin Hartwick path $(c^*(t), k^*(t), r^*(t))$ from (k_0, m_0) satisfying $c^*(t) = \max C(k_0, m_0)$ for all $t \geq 0$.

Any Hartwick path $(c^c(t), k^c(t), r^c(t))$ is interior and egalitarian, satisfies (HaR) for all $t \geq 0$, and has the property that $c^c(t)$, $k^c(t)$ and $r^c(t)$ are continuously differentiable. Hence, by invoking Buchholz, Dasgupta and Mitra (2005, Proposition 3) it follows that it also satisfies (HoR) as a condition for short-run efficiency.

In this section we show that assumptions **A1**–**A3** are sufficient for establishing that any Hartwick path satisfies (CVT). However, while it is clear that any Hartwick path that is not maximin does not satisfy resource exhaustion and is thus inefficient, it is still an open question whether these assumptions imply that the maximin Hartwick path exhausts the resource when time goes to infinity.

We establish that the maximin Hartwick path satisfies resource exhaustion and is efficient under an additional assumption, referred to as **A4**. While imposing this additional assumption is sufficient, we are not able show its necessity. In our analysis of efficiency, we apply the concepts of competitive paths and regular maximin paths (cf. Burmeister and Hammond, 1977; Dixit, Hammond and Hoel, 1980).

A path $(c(t), k(t), r(t))$ from (k_0, m_0) is *competitive* if there exist present-value price functions $p(\cdot) : [0, \infty) \to \mathbb{R}_{++}$ and $(q_1(\cdot), q_2(\cdot)) : [0, \infty) \to \mathbb{R}^2$, where $p(t)$ is continuous and $(q_1(t), q_2(t))$ are differentiable, such that, for all $t \geq 0$,

$$(c(t), k(t), m(t), \dot{k}(t), \dot{m}(t)), \text{ where } \dot{k}(t) = F(k(t), r(t)) - c(t) \text{ and } \dot{m}(t)) = -r(t),$$

maximizes instantaneous profits

$$p(t)c' + q_1(t)\dot{k}' + q_2(t)\dot{m}' + \dot{q}_1(t)k' + \dot{q}_2(t)m'$$

over all quintuples $(c', k', m', \dot{k}', \dot{m}')$ in the production possibility set Y defined by:

$$Y := \{(c, k, m, \dot{k}, \dot{m}) \in \mathbb{R}^3_+ \times \mathbb{R} \times (-\mathbb{R}_+) : c + \dot{k} \leq F(k, (-\dot{m}))\} .$$

Proposition 3 *Assume that F satisfies **A1**–**A3** and that $C(k_0, m_0)$ is non-empty with $c \in C(k_0, m_0)$. Then the Hartwick path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) with $c^c(t) = c$ for all $t \geq 0$ is competitive.*

Proof. Since F is twice continuously differentiable and $k^c(t)$ and $r^c(t)$ are differentiable, we may define

$$p(t) = q_1(t) = \frac{1}{F_2(k^c(t), r^c(t))} \tag{P}$$

$$q_2(t) = 1 \tag{Q}$$

for all $t \geq 0$. Note that, for each $t \geq 0$, $(c^c(t), k^c(t), m^c(t), \dot{k}^c(t), \dot{m}^c(t)) \in Y$. Furthermore, for all $(c', k', m', \dot{k}', \dot{m}') \in Y$, we have by **A2** and **A3**:

$$\begin{aligned} (c' + \dot{k}') - (c^c(t) + \dot{k}^c(t)) &\leq F(k', -\dot{m}') - F(k^c(t), -\dot{m}^c(t)) \\ &\leq F_1(k^c(t), r^c(t)) \left(k' - k^c(t)\right) + F_2(k^c(t), r^c(t)) \left(-\dot{m}' + \dot{m}^c(t)\right) . \end{aligned} \tag{34}$$

Multiplying through (34) by $p(t) > 0$, and using (HoR), (P) and (Q), yields:

$$p(t) \left(c' - c^c(t)\right) + q_1(t) \left(\dot{k}' - \dot{k}^c(t)\right) \leq -\dot{q}_1(t) \left(k' - k^c(t)\right) + q_2(t) \left(-\dot{m}' + \dot{m}^c(t)\right) . \tag{35}$$

Transposing terms in (35) and noting that $\dot{q}_2(t) = 0$ for $t \geq 0$, we obtain

$$\begin{aligned} & p(t)c^c(t) + q_1(t)\dot{k}^c(t) + q_2(t)\dot{m}^c(t) + \dot{q}_1(t)k^c(t) + \dot{q}_2(t)m^c(t) \\ \geq \ & p(t)c' + q_1(t)\dot{k}' + q_2(t)\dot{m}' + \dot{q}_1(t)k' + \dot{q}_2(t)m' \end{aligned}$$

for all $(c', k', m', \dot{k}', \dot{m}') \in Y$ and all $t \geq 0$. ■

Note that only Hotelling's rule is used in this proof. Since, as is easily shown, a competitive path is short-run efficient, it proves the claim that Hotelling's rule is sufficient for short-run efficiency.

To the best of our knowledge the existing literature does not provide a formal proof that the Hartwick path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) with $c^c(t) = c \in C(k_0, m_0)$ for all $t \geq 0$ satisfies the capital value transversality condition. To settle that this is indeed the case, we present here a self-contained treatment, which uses only the maintained assumptions **A1**–**A3**, and the non-emptiness of $C(k_0, m_0)$. No additional assumptions are needed.

Proposition 4 *Assume that F satisfies **A1**–**A3** and that $C(k_0, m_0)$ is non-empty with $c \in C(k_0, m_0)$. Then the Hartwick path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) with $c^c(t) = c$ for all $t \geq 0$ satisfies the capital value transversality condition.*

Proof. Note that $m^c(t)$ is decreasing in t (since $r^c(t) > 0$ for all $t \geq 0$), and bounded below by zero, so it converges to a limit $m^c_\infty := \lim_{t\to\infty} m^c(t) \geq 0$. Note that resource exhaustion is *not* being assumed.

Using (HoR), we know that $\dot{F}_2(k^c(t), r^c(t)) > 0$ for all $t \geq 0$, and so $F_2(k^c(t), r^c(t))$ is increasing for $t \geq 0$. If $F_2(k^c(t), r^c(t))$ were to be bounded above, then there would be a number $B > 0$, such that $F_2(k^c(t), r^c(t)) \leq B$ for all $t \geq 0$. In this case, we would have:

$$\dot{k}^c(t) = F_2(k^c(t), r^c(t))r^c(t) \leq Br^c(t) \text{ for all } t \geq 0\,.$$

However, then by integrating from $t = 0$ to $t = T > 0$, we would get:

$$k^c(T) - k^c(0) \leq B \int_0^T r^c(t)dt \leq Bm_0$$

so that $k^c(t)$ would be bounded above by $k_0 + Bm_0$ for all $t \geq 0$. This would contradict Lemma 5. Thus, $F_2(k^c(t), r^c(t))$ cannot be bounded above and we have:

$$F_2(k^c(t), r^c(t)) \to \infty \quad \text{as } t \to \infty\,.$$

Using (P),

$$p(t) \to 0 \quad \text{as } t \to \infty\,, \tag{36}$$

and (CVT) is equivalent to

$$p(t)k^c(t) \to 0 \quad \text{as } t \to \infty\,. \tag{37}$$

Let $\varepsilon > 0$ be given. By definition of m^c_∞, there is $T_1 > 0$ such that:

$$|m^c(t) - m^c_\infty| < \varepsilon/3 \quad \text{for all } t > T_1\,. \tag{38}$$

Furthermore, because of (36) there is $T_2 > T_1$ such that:

$$p(t)k^c(T_1) < (\varepsilon/3) \quad \text{for all } t > T_2\,. \tag{39}$$

Fix any $T' > T_2$. Then, since $\dot{k}^c(t) > 0$ for all $t \geq 0$, and $p(t)$ is positive and decreasing in t, we can use (HaR) to write:

$$\begin{aligned}\int_{T_1}^{T'} r^c(t)dt = \int_{T_1}^{T'} p(t)\dot{k}^c(t)dt \geq \int_{T_1}^{T'} p(T')\dot{k}^c(t)dt \\ = p(T')\int_{T_1}^{T'} \dot{k}^c(t)dt = p(T')[k^c(T') - k^c(T_1)]\end{aligned} \tag{40}$$

On the other hand, by using (38),

$$\begin{aligned}\int_{T_1}^{T'} r^c(t)dt &= m^c(T_1) - m^c(T') = (m^c(T_1) - m^c_\infty) - (m^c(T') - m^c_\infty) \\ &\leq |m^c(T_1) - m^c_\infty| + |m^c(T') - m^c_\infty| < 2\varepsilon/3\,. \end{aligned} \tag{41}$$

Combining (40) and (41),

$$p(t)[k^c(t) - k^c(T_1)] < 2\varepsilon/3 \quad \text{for all } t > T_2\,.$$

This yields:

$$p(t)k^c(t) < p(t)k^c(T_1) + (2\varepsilon/3) \quad \text{for all } t > T_2\,. \tag{42}$$

Thus, combining (39) and (42), we obtain:

$$p(t)k^c(t) < \varepsilon \quad \text{for all } t > T_2\,.$$

This establishes (37) and thereby (CVT). ■

Propositions 3 and 4 imply that, if $C(k_0, m_0)$ is non-empty, then the Hartwick path from (k_0, m_0) given $c \in C(k_0, m_0)$ solves the minimum resource use problem of minimizing the total resource depletion while sustaining $c(t) \geq c$ indefinitely.

A competitive path $(c(t), k(t), r(t))$ from (k_0, m_0) at present-value prices $p(t)$ and $(q_1(t), q_2(t))$ is a *regular maximin path* if it[9]

(i) is egalitarian,

(ii) satisfies the capital value transversality condition and resource exhaustion,

(iii) has finite consumption value:

$$\int_0^\infty p(t) < \infty\,. \tag{FCV}$$

In view of Propositions 3 and 4, if $C(k_0, m_0)$ is non-empty, the maximin Hartwick path $(c^*(t), k^*(t), r^*(t))$ from (k_0, m_0) satisfying $c^*(t) = \max C(k_0, m_0)$ for all $t \geq 0$ is regular if it satisfies resource exhaustion and has finite consumption value.

As a step towards establishing that the maximin Hartwick path is regular, we first show that any Hartwick path satisfies (FCV) given that the following assumption—entailing that resource input is *important* (Mitra, 1978a, p. 121)—is added.

[9] Dixit, Hammond and Hoel (1980, p. 553) define a regular maximin path through their conditions (a)–(c). In the context of our one-consumption good model, (i) and (iii) correspond to their conditions (a) and (b), while (ii) is equivalent to their condition (c), given that $q_2(t) = 1$ for all $t \geq 0$.

Assumption 4 (A4) $\beta := \inf_{(k,r)\gg 0} F_2(k,r)r/F(k,r) > 0$.

In the Cobb-Douglas case, where the production function is given by (4), $\beta = b$.

Lemma 9 *Assume that F satisfies **A1**–**A4** and that $C(k_0, m_0)$ is non-empty with $c \in C(k_0, m_0)$. Then the Hartwick path $(c^c(t), k^c(t), r^c(t))$ from (k_0, m_0) with $c^c(t) = c$ for all $t \geq 0$ satisfies*

$$\int_0^\infty \frac{1}{F_2(k^c(t), r^c(t))} dt < \infty\,. \tag{43}$$

Proof. By invoking **A4**, for all $T > 0$ we have:

$$\int_0^T \frac{1}{F_2(k^c(t), r^c(t))} dt = \int_0^T \frac{r^c(t)}{F_2(k^c(t), r^c(t))r^c(t)} dt \leq \int_0^T \frac{1}{\beta F(k^c(t), r^c(t))} r^c(t) dt$$
$$< \int_0^T \frac{1}{\beta c} r^c(t) dt = \frac{1}{\beta c} \int_0^T r^c(t) dt \leq \frac{m_0}{\beta c}\,.$$

since $\dot{k}^c(t) = F(k^c(t), r^c(t)) - c > 0$, so that $F(k^c(t), r^c(t)) > c$, for all $t \geq 0$. The integral is increasing in T and bounded above by $(m_0/\beta c)$, so it converges as $T \to \infty$. Thus (43) follows. ∎

Observe that

$$\int_0^\infty \frac{1}{F_2(k^c(t), r^c(t))} dt$$

is the marginal resource cost of a uniform increment to consumption, given the initial capital stock k_0. Therefore, it follows from Lemma 9 that under **A1**–**A4** any unused resource can be translated into a uniform addition to consumption provided that there it is feasible to sustain a path from (k_0, m_0) with consumption bounded away from zero. Hence, the maximin Hartwick path satisfies resource exhaustion, and is thus regular. The following theorem turns this intuition into a formal argument.

Theorem 4 *Assume that F satisfies **A1**–**A4** and that $C(k_0, m_0)$ is non-empty. Then the maximin Hartwick path $(c^*(t),\ k^*(t), r^*(t))$ from (k_0, m_0) satisfying $c^*(t) = \max C(k_0, m_0)$ for all $t \geq 0$ is regular.*

Proof. By Propositions 3 and 4 and Lemma 9, it remains to be shown that

$$m^*(t) \to 0 \quad \text{as } t \to \infty\,. \tag{44}$$

Suppose (44) did not hold. Then:

$$m^*(t) \to m^*_\infty > 0 \quad \text{as } t \to \infty\,.$$

Denote $[m_0/(m_0 - m^*_\infty)]$ by θ. Then $\theta > 1$. Define:

$$(c(t), k(t), r(t)) = (F(k^*(t), \theta r^*(t)) - \dot{k}^*(t), k^*(t), \theta r^*(t)) \quad \text{for all } t \geq 0\,.$$

Note that $r(t) > 0$ for $t \geq 0$, $r(t)$ is continuous on $[0, \infty)$, and

$$\int_0^\infty r(t)dt = \theta \int_0^\infty r^c(t)dt = \theta(m_0 - m^*_\infty) = m_0\,.$$

Also, $k(t) = k^*(t)$ is continuously differentiable on $[0, \infty)$, so $c(t) = F(k(t), r(t)) - \dot{k}(t)$ is continuous on $[0, \infty)$. Finally, for $t \geq 0$, we have:

$$\begin{aligned}
c(t) &= F(k(t), r(t)) - \dot{k}(t) = F(k^*(t), \theta r^*(t)) - \dot{k}^*(t) \\
&= F(k^*(t), \theta r^*(t)) - F(k^*(t), r^*(t)) + F(k^*(t), r^*(t)) - \dot{k}^*(t) \\
&= F(k^*(t), \theta r^*(t)) - F(k^*(t), r^*(t)) + c \geq F_2(k^*(t), \theta r^*(t))(\theta - 1)r^*(t) + c \\
&= [(\theta - 1)/\theta]F_2(k^*(t), \theta r^*(t))\theta r^*(t) + c \geq [(\theta - 1)/\theta]\beta F(k^*(t), \theta r^*(t)) + c \\
&\geq [(\theta - 1)/\theta]\beta F(k^*(t), r^*(t)) + c \geq [(\theta - 1)/\theta]\beta c + c\,.
\end{aligned}$$

This shows that $(c(t), k(t), r(t))$ is a path from (k_0, m_0), and

$$c(t) \geq c\{1 + [(\theta - 1)/\theta]\beta\} > c\,.$$

But, then, $(k^*(t), r^*(t), c^*(t))$ is not a maximin path from (k_0, m_0), a contradiction, Thus, (44) must hold, and $(k^*(t), r^*(t), c^*(t))$ is a regular maximin path. ■

9 Concluding remarks

We have provided a complete technological characterization of the sustainability problem and established general existence of a maximin optimal path in the Dasgupta-Heal-Solow-Stiglitz model of capital accumulation and resource depletion under weaker conditions that those employed in previous work. Though our method of proof, we have also offered new insights into the meaning and significance of Hartwick's reinvestment rule.

Unfortunately, our constructive approach may not extend to different criteria and environments and thus the prospects for wider applicability of our strategy of proof might be limited.

(a) With another objective function than maximin it is not necessarily the case that the ratio of capital accumulation to resource depletion is maximized. E.g.

under classical utilitarianism, capital is accumulated faster than this, to meet the demands of higher consumption in the future (see Dasgupta and Heal, 1979; Asheim, Buchholz, Hartwick, Mitra and Withagen, 2007).

(b) In the DHSS model with two capital goods, one obtains Buchholz, Dasgupta and Mitra's (2005, Proposition 3) result that constant consumption and Hartwick's rule implies Hotelling's rule as a condition for short-run efficiency. As mentioned above, this property is utilized in this paper, since short-run efficiency follows from constant consumption and maximization of the ratio of capital accumulation to resource depletion. In models with more than two capital goods, conditions for short-run efficiency must be invoked directly.

(c) The strategy of proof with its phase diagram argument relies on a stationary environment where the future development of the optimal path depends on the vector of stocks only. It would not directly extend to models with technological progress or population growth.

A Appendix: An example without maximin existence

Consider the case where F is given by (10) and $(k_0, m_0) = (1, 1)$.

Sustainability. We first claim that $C(k_0, m_0)$ is non-empty. To establish this, simply define:

$$k(t) = 1, \quad r(t) = 0, \quad c(t) = 1 \quad \text{for all } t \geq 0\,,$$

and note that $\dot{k}(t) = 0$ for $t \geq 0$. Thus, $(c(t), k(t), r(t))$ is a path from $(k_0, m_0) = (1, 1)$, and $c(t) = 1 > 0$ for all $t \geq 0$. This establishes our claim.

An upper bound on sustainable consumption. We now claim that that there is no path $(c(t), k(t), r(t))$ satisfying:

$$c(t) \geq 2 \quad \text{for all } t \geq 0\,. \tag{A1}$$

Suppose, there were such a path. We then establish the following steps.

Step 1: We must have $k(t) < 2$ *for all* $t \geq 0$. For if $k(t) \geq 2$ for some $t \geq 0$, then we can define $T = \inf\{t \geq 0 : k(t) \geq 2\}$. By continuity of $k(t)$, we must have $k(T) = 2$. Since $k(0) = 1$, we know that $T > 0$. Furthermore,

$$k(t) < 2 \quad \text{for all } t \in [0, T)\,. \tag{A2}$$

Denote $(2-k(t))$ by $\alpha(t)$ for all $t \in [0,T]$, and let $\sigma := \int_0^{T/2} \alpha(t)dt$. Then, by (A2), $\sigma > 0$, and using the fact that for all $t \in [0,T)$,

$$\dot{k}(t) = k(t) + r(t) - c(t) \leq k(t) + r(t) - 2 = r(t) - \alpha(t)\,,$$

we obtain for all $\tau \in [T/2, T)$,

$$k(\tau) - k(0) = \int_0^\tau \dot{k}(t)dt = \int_0^\tau r(t)dt - \int_0^\tau \alpha(t)dt \leq \int_0^\tau r(t)dt - \int_0^{T/2} \alpha(t)dt \leq m(0) - \sigma = 1 - \sigma.$$

Thus, for all $\tau \in [T/2, T)$, we get $k(\tau) \leq k(0) + (1-\sigma) = (2-\sigma)$. So by continuity of $k(t)$, we obtain $k(T) \leq 2-\sigma$, and this contradicts the fact that $k(T) = 2$. This completes Step 1.

Define $\alpha(t) = 2 - k(t)$ for all $t \geq 0$. Then, $\alpha(t) > 0$ for all $t \geq 0$ by Step 1, and therefore:

$$\beta := \int_0^1 \alpha(t)dt > 0\,.$$

Step 2: We must have $k(t) \leq 2 - \beta$ for all $t \geq 1$. To see this, note that for all $t \geq 0$,

$$\dot{k}(t) = k(t) + r(t) - c(t) \leq k(t) + r(t) - 2 = r(t) - \alpha(t)\,.$$

so that for all $T \geq 1$,

$$k(T) - k(0) = \int_0^T \dot{k}(t)dt = \int_0^T r(t)dt - \int_0^T \alpha(t)dt \leq \int_0^T r(t)dt - \int_0^1 \alpha(t)dt \leq m(0) - \beta = 1 - \beta,$$

and consequently, $k(T) \leq k(0) + (1-\beta) = 2 - \beta$ for all $T \geq 1$. This completes Step 2.

Step 3: $k(t) < 0$ for all $t > (2+\beta)/\beta$. For all $t \geq 0$, we have $\dot{k}(t) = k(t) + r(t) - c(t)$, and we can write for all $T > 1$,

$$\begin{aligned} k(T) - k(0) &= \int_0^T \dot{k}(t)dt = \int_0^T k(t)dt + \int_0^T r(t)dt - \int_0^T c(t)dt \\ &\leq \int_0^1 k(t)dt + \int_1^T k(t)dt + \int_0^T r(t)dt - 2T \\ &< 2 + (2-\beta)(T-1) + 1 - 2T = 1 - \beta(T-1)\,, \end{aligned} \tag{A3}$$

the third line of (A3) following from Steps 1 and 2. Thus, $k(T) \leq k(0) + 1 - \beta(T-1) = 2 - \beta(T-1)$ for all $T > 1$. For $T > (2+\beta)/\beta$, we have $(T-1) > 2/\beta$, and so $\beta(T-1) > 2$. Thus, for $T > (2+\beta)/\beta$, $k(T) \leq 2 - \beta(T-1) < 0$ and this establishes Step 3.

By Step 3, the hypothesis that there is a path $(c(t), k(t), r(t))$ satisfying (A1) must be false, and this establishes our claim.

We have now demonstrated that an upper bound of $C(k_0, m_0)$ is 2. We will show in the next section that this is also its least upper bound.

The supremum of $C(k_0, m_0)$. We now show that, given any $\varepsilon \in (0,1)$, there is a path $(c(t), k(t), r(t))$ satisfying:

$$c(t) \geq 2 - \varepsilon \quad \text{for all } t \geq 0\,.$$

Given the ε, define:

$$n = (1/\varepsilon) \quad \text{and} \quad T = 2/(n+1)^3\,, \tag{A4}$$

and determine the path of resource depletion by:

$$r(t) = \begin{cases} \dfrac{2}{T} - \dfrac{2t}{T^2} & \text{for } t \in [0,T]\,, \\ 0 & \text{for } t > T\,. \end{cases}$$

Then, $r(t) \geq 0$ for $t \in [0,T]$, with $r(T) = 0$, and $r(t) \to 0$ as $t \to T$. Thus $r(t)$ is continuous for $t \geq 0$. Furthermore,

$$\int_0^\infty r(t)dt = \int_0^T r(t)dt = \int_0^T \left\{\frac{2}{T} - \frac{2t}{T^2}\right\} dt = 2 - \frac{2}{T^2}\left[\frac{t^2}{2}\right]_0^T = 2 - 1 = 1 = m_0\,.$$

Denote $(1-\varepsilon^2)$ by λ, and determine the capital path by:

$$k(t) = \begin{cases} 1 + \lambda\left[\dfrac{2t}{T} - \dfrac{t^2}{T^2}\right] & \text{for } t \in [0,T]\,, \\ 1+\lambda & \text{for } t > T\,. \end{cases} \tag{A5}$$

Note that $k(0) = 1 = k_0$ and $k(t) \geq 1$ for all $t \geq 0$. Furthermore,

$$\dot{k}(t) = \lambda\left[\frac{2}{T} - \frac{2t}{T^2}\right] = \lambda r(t) \geq 0 \quad \text{for } t \in [0,T)$$

with $\dot{k}(T-) = 0 = \dot{k}(T+)$. Thus, $k(t)$ is a C^1 function on $\mathbb{R}_+$, and $\dot{k}(t) = \lambda r(t)$ for all $t \geq 0$.

If we now determine the consumption path by:

$$c(t) = (1-\lambda)r(t) + k(t) \quad \text{for all } t \geq 0\,, \tag{A6}$$

then clearly $c(t) \geq 0$ for $t \geq 0$, and $c(t) = F(k(t), r(t)) - \lambda r(t) = F(k(t), r(t)) - \dot{k}(t)$ for $t \geq 0$. So $\{c(t), k(t), r(t)\}$ is a path from (k_0, m_0).

It remains to show that $c(t) \geq 2 - \varepsilon$ for all $t \geq 0$. For $t > T$, we have $r(t) = 0$ and $k(t) = 1 + \lambda$, so by (A5) and (A6),

$$c(t) = k(t) = 1 + \lambda = 2 - \varepsilon^2 > 2 - \varepsilon \quad \text{for } t > T\,. \tag{A7}$$

So, we now concentrate on $t \in [0,T]$.

Define $N = nT/(n+1)$ Then,

$$\begin{aligned} k(N) &= 1 + \lambda\left[\frac{2N}{T} - \frac{N^2}{T^2}\right] = 1 + \lambda\left[\frac{2n}{(n+1)} - \frac{n^2}{(n+1)^2}\right] \\ &= 1 + \frac{\lambda n}{(n+1)}\left[2 - \frac{n}{(n+1)}\right] = 1 + \frac{\lambda n}{(n+1)}\frac{(n+2)}{(n+1)} = 1 + \lambda\left[1 - \frac{1}{(n+1)^2}\right]. \end{aligned} \tag{A8}$$

Now, by choice of n in (A4), we have:

$$(n+1)^2 \geq (n+1) = \frac{1}{\varepsilon} + 1 = \frac{1+\varepsilon}{\varepsilon}$$

and so:

$$\frac{1}{(n+1)^2} \leq \frac{\varepsilon}{1+\varepsilon}\,.$$

Thus,

$$\left[1 - \frac{1}{(n+1)^2}\right] \geq \left[1 - \frac{\varepsilon}{1+\varepsilon}\right] = \frac{1}{1+\varepsilon}\,. \tag{A9}$$

Using (A9) in (A8), we get:

$$k(N) \geq 1 + \frac{\lambda}{1+\varepsilon} = 1 + \frac{1-\varepsilon^2}{1+\varepsilon} = 2 - \varepsilon\,.$$

Since $\dot{k}(t) \geq 0$ for all $t \geq 0$, we have $k(t) \geq 2 - \varepsilon$ for all $t \in [N, T]$ Consequently, using (A6),

$$c(t) = (1-\lambda)r(t) + k(t) \geq k(t) \geq 2 - \varepsilon \quad \text{for all } t \in [N, T]\,. \tag{A10}$$

Finally, we turn to $t \in [0, N)$. Here, we have by (A6),

$$\begin{aligned} c(t) &= (1-\lambda)r(t) + k(t) \geq (1-\lambda)r(t) + 1 = \varepsilon^2 r(t) + 1 = \varepsilon^2\left[\frac{2}{T} - \frac{2t}{T^2}\right] + 1 \\ &\geq \varepsilon^2\left[\frac{2}{T} - \frac{2N}{T^2}\right] + 1 = \varepsilon^2\left[\frac{2}{T} - \frac{2nT}{(n+1)T^2}\right] + 1 \\ &= \frac{2\varepsilon^2}{T}\left[1 - \frac{n}{(n+1)}\right] + 1 = \frac{2\varepsilon^2}{(n+1)T} + 1 = \varepsilon^2(n+1)^2 + 1 > \varepsilon^2 n^2 + 1 = 2\,, \end{aligned} \tag{A11}$$

the second line of (A11) following from the definition of N, and the last line of (A11) following from the definitions of T and n in (A4).

Combining (A7), (A10) and (A11), we have $c(t) \geq 2 - \varepsilon$ for all $t \geq 0$, and so:

$$\inf_{t \geq 0}\ c(t) \geq 2 - \varepsilon\,.$$

Combining the result of this part with the previous one, we conclude that the supremum of $C(k_0, m_0)$ is equal to 2. However, as shown in the previous part, there is no path in $C(k_0, m_0)$ which attains this supremum. Thus, there is no maximin path in this model.

It is of interest to note that in discussing basically the same example of the production function, Dasgupta and Heal (1974, p. 18) claim that there exists an optimal path in this case. In fact they make the claim that the Dirac delta function is the optimal strategy for optimal depletion of the resource. However, given that the integral of resource depletion needs to be well-defined, at least as a Lebesgue integral, it follows that we cannot admit the Dirac delta-function as a feasible path of resource depletion.

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